Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 5. Common Continuous Distribution

5.3 Gamma Distribution

GuanNan Wang
gwang01@wm.edu

Gamma Distribution

Gamma Distribution

A continuous random variable X with pdf

$$f(x) = \frac{\lambda^{\kappa} x^{\kappa-1} e^{-\lambda x}}{\Gamma(\kappa)}, \quad x > 0$$

for some real constants $\lambda > 0$ and $\kappa > 0$ is a $\mathrm{Gamma}(\lambda, \kappa)$ random variable.

Mean

Variance

MGF

R Functions

R Functions

Function	Returned Value
dgamma(x, κ, λ)	calculates the probability density function $f(x)$
pgamma(x, κ, λ)	calculates the cumulative distribution function $F(x)$
qgamma(u, κ, λ)	calculates the percentile (quantile) $F^{-1}(u)$
rgamma(m, κ, λ)	generates m random variates

Special Cases

- **Special Case 1:** Exponential distribution when $\kappa = 1$

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0.$$

$$E(X) = \frac{1}{\lambda}, \quad V(X) = \frac{1}{\lambda^2}, \quad E\left\{\left(\frac{X-\mu}{\sigma}\right)^3\right\} = 2, \quad E\left\{\left(\frac{X-\mu}{\sigma}\right)^4\right\} = 9.$$

- **Special Case 2:** Erlang distribution when κ is a positive integer k

$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}, \quad x > 0$$

$$E(X) = \frac{k}{\lambda}, \; V(X) = \frac{k}{\lambda^2}, \; E\left\{\left(\frac{X-\mu}{\sigma}\right)^3\right\} = \frac{2}{\sqrt{k}},$$

$$E\left\{\left(\frac{X-\mu}{\sigma}\right)^4\right\} = \frac{3(k+2)}{k}.$$

Special Cases

- **Special Case 3:** χ^2 distribution when κ is $k/2$, where k is a parameter known as the degrees of freedom, and $\lambda = 1/2$,

$$f(x) = \frac{x^{k/2-1}e^{-x/2}}{\Gamma(k/2)2^{k/2}}, \quad x > 0$$

$$E(X) = k,\ V(X) = 2k,\ E\left\{\left(\frac{X-\mu}{\sigma}\right)^3\right\} = \frac{2\sqrt{2}}{\sqrt{k}},$$

$$E\left\{\left(\frac{X-\mu}{\sigma}\right)^4\right\} = \frac{3(k+4)}{k}.$$

Example 1

Suppose $X_1,\ X_2,\ \cdots,\ X_K$ are independent random variables with pdf $f(x) = \lambda e^{-\lambda x},\ \lambda > 0,\ x > 0$, let $Y = \sum_{i=1}^{K} X_i$, what is the distribution of Y?

Summary

- Gamma distribution is a generalization of the Exponential distribution
- Probability density function

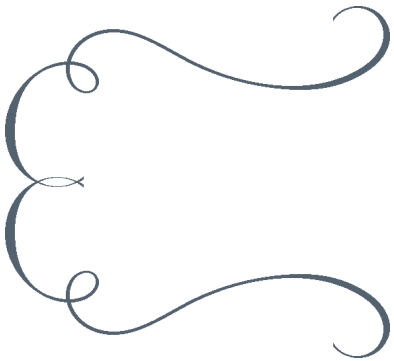
$$f(x) = \frac{\lambda^{\kappa} x^{\kappa-1} e^{-\lambda x}}{\Gamma(\kappa)}, \quad x > 0$$

- Positive scale parameter λ
- Positive shape parameter κ
- Population mean and variance

$$\mu = \frac{\kappa}{\lambda} \quad \text{and} \quad \sigma^2 = \frac{\kappa}{\lambda^2}$$

- Important special cases
  - Exponential distribution when $\kappa = 1$
  - Erlang distribution when κ is a positive integer k
  - χ^2 distribution when $\lambda = 1/2$ and $\kappa = k/2$

Thank You



THANK YOU!