Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 3. Random Variables

3.3 Cumulative Distribution Function

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Cumulative Distribution Functions

Cumulative Distribution Functions (CDF)

- **Distribution of a discrete random variable** X is characterized by its probability mass function $f(x)$ and its associated support $\mathcal{A}$.
- **Distribution of a continuous random variable** X is characterized by its probability density function $f(x)$ and its associated support $\mathcal{A}$.
- **The cumulative distribution function (cdf)** applies to both types of random variables $F(x) = P(X \leq x)$.
  - **Discrete** random variable X

$$F(x) = P(X \leq x) = \sum_{w \leq x} f(w).$$

  - **Continuous** random variable X

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(w)dw.$$

Properties of CDF

- Since $F(x)$ is defined as a probability, $0 \leq F(x) \leq 1$.
- $F(x)$ is a nondecreasing function of x, that is, for $a < b$, $F(a) \leq F(b)$.
- $\lim_{x \to -\infty} F(x) = 0$.
- $\lim_{x \to \infty} F(x) = 1$.
- $P(a < X \leq b) = F(b) - F(a) = P(X \leq b) - P(X \leq a)$.
- The random variables X and Y are identically distributed if and only if they have identical cumulative distribution functions.

Example 1

Flip a fair coin twice. Let X be the number of heads tossed. Find $F(x)$.

Example 2

Find the cumulative distribution function for a continuous random variable X that is uniformly distributed between 0 and 1.

More properties of CDF

- If X is a discrete random variable, $F(x)$ is a right-continuous step function; if X is a continuous random variable, $F(x)$ is a continuous function.
- For a discrete random variable X with support $\mathcal{A} = \{x_1, x_2, \ldots, x_n\}$, where $x_1 < x_2 < \cdots < x_n$,

$$f_X(x_i) = F_X(x_i) - F_x(x_{i-1}),\ i = 2, 3, \ldots, n,$$

  and $f_X(x_1) = F(x_1)$.
- For a continuous random variable X, if $f_X(x)$ is continuous, then $F'_X(x)$ exists, and

$$f_X(x) = F'_X s(x).$$

- In actuarial science, biostatistics, and reliability, the related **survivor** function, defined as $S(x) = P(X \leq x)$, is used more frequently than the cumulative distributed function. If X is a patient's survival time after a particular type of surgery, then $S(x)$ can be considered as the probability that the patient **survives** to time x.

Example 3

Find the PMF of the random variable X with CDF given by

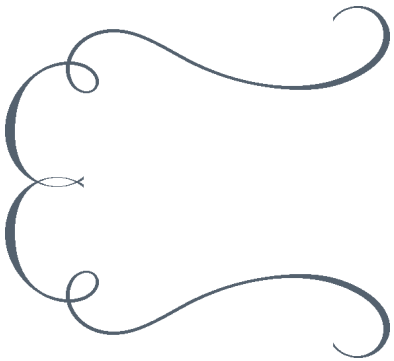
$$F_X(x) = \begin{cases} 1/9 & x = 1 \\ 3/9 & x = 2 \\ 5/9 & x = 3 \\ 6/9 & x = 4 \\ 8/9 & x = 6 \\ 1 & x = 9 \end{cases}$$

Example 4

Find the PDF of the random variable X with CDF given by

$$F_X(x) = \frac{x^2}{4}, \quad 0 < x < 2$$

Thank You



THANK YOU!