Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 2. Probability

2.2 Probability Axioms

GuanNan Wang
gwang01@wm.edu

Probability Axioms

Relative Frequency

Perform a random experiment n times. Let x be the number of times that the event A occurs. The ratio x/n is the **relative frequency** of the event A in the n experiment.

Estimate the probability that a coin comes up “heads" on a single toss of a fair coin.

Limiting Relative Frequency

Limiting Relative Frequency

The **limiting relative frequency** of the event A

$$P(A) = \lim_{n \to \infty} \frac{x}{n}$$

is called the

- probability that the outcome of the random experiment is in A, or
- probability of event A,

if the limit exists.

Classical Approach

If the sample space S consists of equally-likely outcomes, then the probability oof event A is the ratio of the number of elements in A to the number of elements in S:

$$P(A) = \frac{N(A)}{N(S)}.$$

Toss a fair cooin three times and observed sequence of heads and tails. Find the probability that all three ourcomes are heads.

Kolmogorov Axioms

Kolmogorov Axioms

Consider a random experiment with sample space S and an event $A \subset S$ of interest. If $P(A)$ is defined and

- **Axiom 1.** $P(A) \geq 0$,
- **Axiom 2.** $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$ where $A_1, A_2, \ldots$ are disjoint events, and
- **Axiom 3.** $P(S) = 1$,

then $P(A)$ is the probability of event A occurring.

Complementary Probability

For each $A \subset S$ $P(A) = 1 - P(A^c)$.

Theorems

Theorem 2.2

If $A_1, A_2 \subset S$ such that $A_1 \subset A_2$ then $P(A_1^c \cap A_2) = P(A_2) - P(A_1)$, and therefore $P(A_1) \leq P(A_2)$.

Theorem 2.3

$P(\Phi) = 0$.

Theorem 2.4

For every $A \subset S$, $0 \leq P(A) \leq 1$.

Theorems

Theorem 2.5

(Addition Rule) If $A_1, A_2 \subset S$ then

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2).$$

Theorems

If events $A_1, A_2, \ldots, A_n$ are disjoint andd their union is the sample space S, then they form a partition of S.

$$A_i \cap A_j = \Phi \text{ for } i \neq j \qquad \text{and} \qquad S = \cup_{i=1}^{n} A_i$$

Theorem 2.6

For events $A_1, A_2, \ldots, A_n$ that form a partition of S and another event $B \subset S$,

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i).$$

Example 2

Considder the events A_1 and A_2 with associated probabilities $P(A_1) = 0.3$, $P(A_2) = 0.5$, and $P(A_1 \cup A_2) = 0.6$. Find $P(A_1 \cap A_2)$ and $P(A_1 \cap A_2^c)$.

Example 3

Roll a pair of fair dice. Find $P(A_1), P(A_2), \ldots, P(A_5)$ for the events

A_1: rolling a total of 7
A_2: rolling a total of 12
A_3: rolling doubles
A_4: rolling 5, either individually or as a total
A_5: rolling numbers that differ by 3

THANK YOU!