Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 6. Joint Distribution

6.3 Joint Moment Generating Functions

GuanNan Wang
gwang01@wm.edu

Joint Moment Generating Functions

Let (X_1, X_2) be a random vector. The **joint moment generating function** of (X_1, X_2) is

$$M(t_1, t_2) = E(e^{t_1 X_1 + t_2 X_2})$$

provided that the expected value exists on $-h_1 < t_1 < h_1$ and $-h_2 < t_2 < h_2$ for some positive constants h_1 and h_2.

- $E(X_1) = \left.\frac{\partial M(t_1,t_2)}{\partial t_1}\right|_{(t_1,t_2)=(0,0)}$, $\quad E(X_2) = \left.\frac{\partial M(t_1,t_2)}{\partial t_2}\right|_{(t_1,t_2)=(0,0)}$
- $E(X_1^2) = \left.\frac{\partial^2 M(t_1,t_2)}{\partial t_1^2}\right|_{(t_1,t_2)=(0,0)}$, $\quad E(X_2^2) = \left.\frac{\partial^2 M(t_1,t_2)}{\partial t_2^2}\right|_{(t_1,t_2)=(0,0)}$
- $E(X_1 X_2) = \left.\frac{\partial^2 M(t_1,t_2)}{\partial t_1 \partial t_2}\right|_{(t_1,t_2)=(0,0)}$
- Marginal moment generating functions: $M_{X_1}(t_1) = E\left(e^{t_1 X_1}\right) = M(t_1, 0)$, $\quad M_{X_2}(t_2) = E\left(e^{t_2 X_2}\right) = M(0, t_2)$
- The random variables X_1 and X_2 are independent iff
$$M(t_1, t_2) = M_{X_1}(t_1) M_{X_2}(t_2)$$

Example 1

Let X_1 and X_2 be continuous random variables whose distrirbution is described by the joint probability density function

$$f(x_1, x_2) = \lambda_1 e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2}, \quad x_1 > 0,\ x_2 > 0$$

for real positive parameters λ_1 and λ_2. Find the joint moment generating function for X_1 and X_2 and use it to find $E(X_1 X_2)$.

$$M(t_1, t_2) = E\Big(\underbrace{e^{t_1X_1 + t_2X_2}}_{g(X_1, X_2)}\Big) = \int_0^\infty \int_0^\infty e^{t_1x_1 + t_2x_2} f(x_1, x_2)\, dx_2\, dx_1$$

$$= \int_0^\infty \int_0^\infty e^{t_1x_1} e^{t_2x_2} \lambda e^{-\lambda_1 x_1} \lambda_2 e^{-\lambda_2 x_2}\, dx_2\, dx_1$$

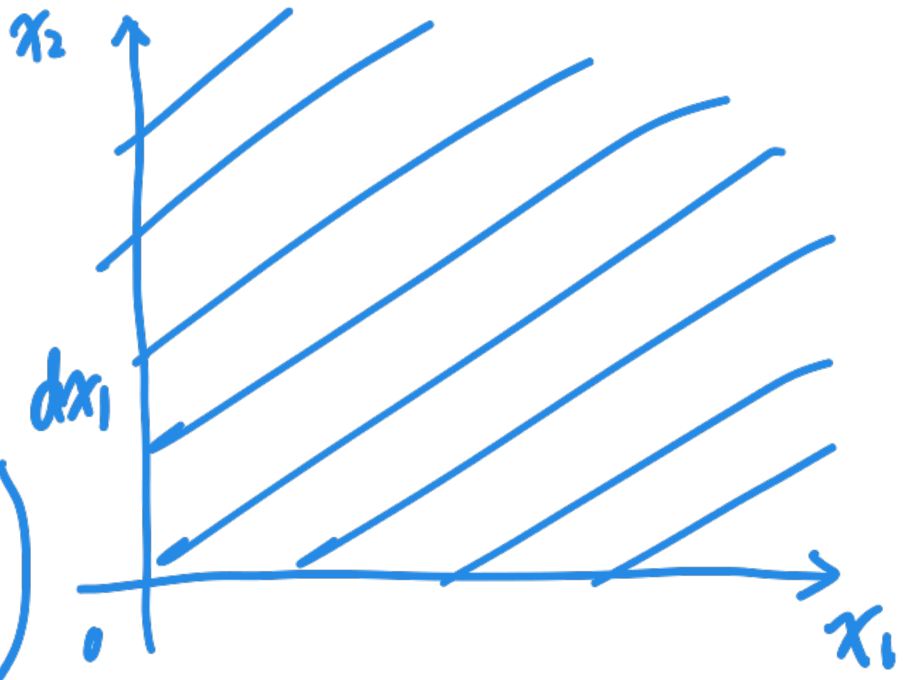
$$= \int_0^\infty e^{t_1x_1} \lambda_1 e^{-\lambda_1 x_1} \int_0^\infty \lambda_2 e^{-(\lambda_2 - t_2)x_2}\, dx_2\, dx_1$$

$t_1 < \lambda_1$

$t_2 < \lambda_2$

$$= \int_0^\infty \lambda_1 e^{-(\lambda_1 - t_1)x_1}\, dx_1 \left(-\frac{\lambda_2 e^{-(\lambda_2 - t_2)x_2}}{\lambda_2 - t_2} \Bigg|_0^\infty \right)$$

$$= \frac{\lambda_1}{\lambda_1 - t_1} \frac{\lambda_2}{\lambda_2 - t_2}$$



$$E(X_1 X_2) = \frac{\partial^2 M(t_1, t_2)}{\partial t_1 \partial t_2} \Bigg|_{t_1=0,\, t_2=0}$$

$$= \frac{\partial^2}{\partial t_1 \partial t_2} \left(\frac{\lambda_1}{\underset{\sim\sim\sim}{\lambda_1 - t_1}} \frac{\lambda_2}{\lambda_2 - t_2} \right) \Bigg|_{t_1=0,\, t_2=0}$$

$$= \frac{+\lambda_1}{(\lambda_1 - t_1)^2} * \frac{\lambda_2}{(\lambda_2 - t_2)^2} \Bigg|_{t_1=0,\, t_2=0}$$

$$= \frac{\lambda_1}{\lambda_1^2} \frac{\lambda_2}{\lambda_2^2} = \frac{1}{\lambda_1 \lambda_2}$$

THANK YOU!