Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 6. Joint Distribution

6.3 Expected Values

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Example 1

For the random variables X and Y with joint probability density function

$$f(x, y) = 2, \quad x > 0, \ y > 0, \ x + y < 1,$$

find $V(E(Y|X))$.

$E(E(Y|X))$

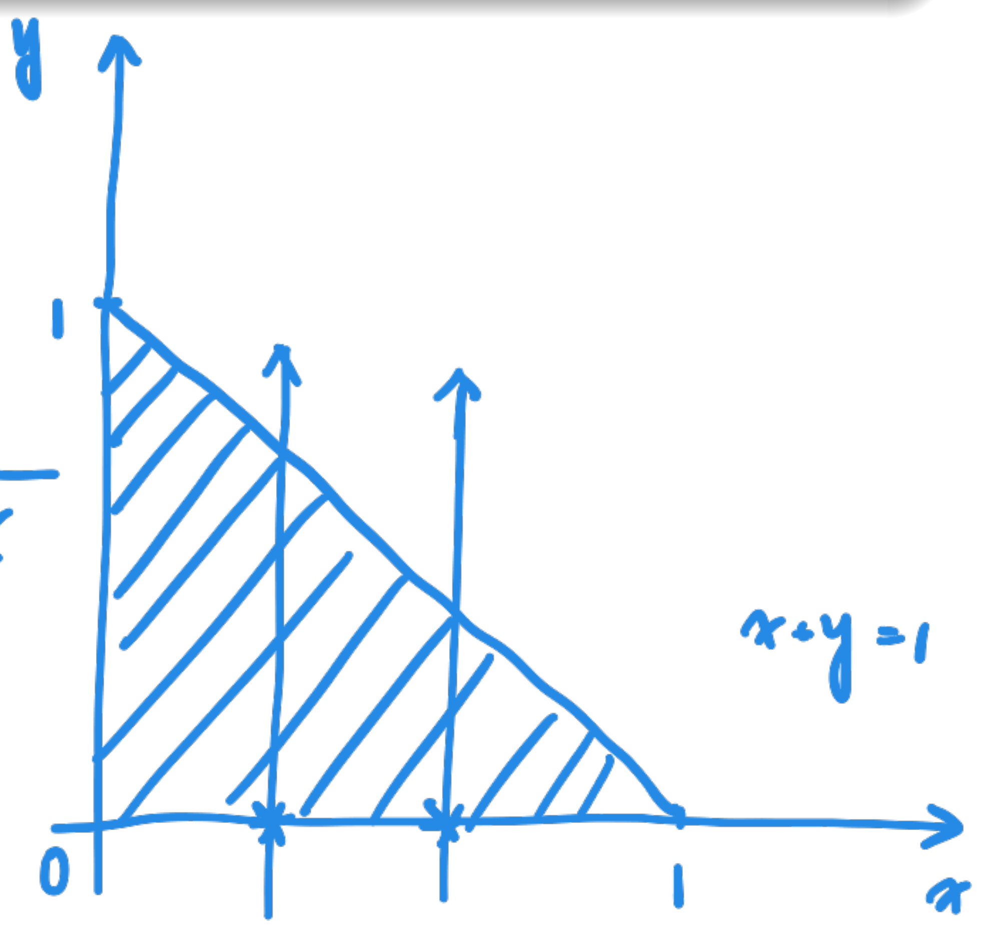
$$f_X(x) = \int_0^{1-x} f(x,y)\,dy = \int_0^{1-x} 2\,dy$$

$$= 2 - 2x, \quad 0 < x < 1$$

$$f_{Y|X=x}(y|X=x) = \frac{f(x,y)}{f_X(x)} = \frac{2}{2-2x} = \frac{1}{1-x}$$

$$0 < y < 1-x; \quad 0 < x < 1$$

$$E(Y|x) = \int_0^{1-x} y \frac{1}{1-x}\,dy = \frac{1-x}{2}, \quad 0 < x < 1$$



$$V\{E(Y|X)\} = V\left(\frac{1-X}{2}\right) = V\left(\frac{1}{2} - \frac{X}{2}\right) = V\left(\ominus\frac{X}{2}\right) = V\left(\frac{X}{2}\right)$$

$$= \frac{1}{4}V(X) = \frac{1}{4}\times\frac{1}{18} = \frac{1}{72}$$

$$E(X) = \int_0^1 x f_X(x)\,dx = \int_0^1 x(2-2x)\,dx = \frac{1}{3}$$

$$E(X^2) = \int_0^1 x^2 f_X(x)\,dx = \int_0^1 x^2(2-2x)\,dx = \frac{1}{6}$$

$$V(X) = E(X^2) - \{E(X)\}^2 = \frac{1}{6} - \frac{1}{9} = \frac{1}{18}$$

$$E\{E(Y|X)\} = E\left(\frac{1-X}{2}\right) = E\left(\frac{1}{2} - \frac{X}{2}\right) = \frac{1}{2} - \frac{1}{2}E(X) = \frac{1}{2} - \frac{1}{2}\times\frac{1}{3}$$

$$= \frac{1}{2} - \frac{1}{6} = \frac{1}{3}$$

$E(Y) = ?$ $\quad f_Y(y) = \int_0^{1-y} 2\,dx = 2-2y, \quad 0<y<1$

$$E(Y) = \int_0^1 y f_Y(y)\,dy = \int_0^1 y(2-2y)\,dy = \frac{1}{3}$$

$E(Y) = E\{E(Y|X)\}$

Theorem 6.11

If X and Y are random variables, then

$$E(X) = E\{E(X|Y)\}$$

when the expectations exist.

Proof: $E(X) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x f(x,y)\,dx\,dy$

$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x\, f_{X|Y=y}(x|Y=y)\, f_Y(y)\,dx\,dy$

$= \int_{-\infty}^{\infty} E(X|Y=y)\, f_Y(y)\,dy$

$= E\{E(X|Y)\}$

$V(E(X|Y)) \leq V(X)$

Theorem 6.12

If X and Y are random variables, then

$$V(X) = E\{V(X|Y)\} + V\{E(X|Y)\}$$

when the expectations exist.

Proof. $E\{V(X|Y)\} = E\{E(X^2|Y) - \{E(X|Y)\}^2\}$

$$= E\{E(X^2|Y)\} - E[\{E(X|Y)\}^2]$$

$$= E(X^2) - E[\{E(X|Y)\}^2]$$

$$= E(X^2) - \{E(X)\}^2 - E[\{E(X|Y)\}^2] + \{E(X)\}^2$$

$$= V(X) - E[\{E(X|Y)\}^2] + [E\{E(X|Y)\}]^2$$

$$= V(X) - V(E(X|Y))$$

Example 2

The binomial distribution was introduced asthe number of successes in n independent Bernoulli trials, each with a probability of success p. The population mean and variance of $X \sim Bin(n, p)$ are

$$E(X) = np \quad \text{and} \quad V(X) = np(1 - p).$$

But what if an application arises where n is fixed but p is unknown? Such an application might arise in an area of quality control known as **acceptance sampling**, where a fixed number of items are sampled from a large lot. To be more specific, assume that p is no longer a fixed probability, but is itself a random variable. Furthermore, assume that you have so little information about the probability of success on each Bernoulli trial that you can only state that the probability of success P is a random variable that is equally likely to fall between 0 and 1. That is to say, $P \sim U(0, 1)$. The problem here is to find $E(X)$ and $V(X)$ when $X \sim Bin(n, P)$, where $P \sim U(0, 1)$.

$$P \sim U(0,1) \Rightarrow \begin{aligned} E(P) &= \tfrac{1}{2} \\ V(P) &= \tfrac{1}{12} \end{aligned}$$

$$X \sim \text{Bin}(n, P) \Rightarrow \begin{aligned} E(X|P) &= nP \\ V(X|P) &= nP(1-P) \end{aligned}$$

$$E(X) = E\{E(X|P)\} = E\{nP\} = n\,E(P) = \frac{n}{2}$$

$$\begin{aligned} V(X) &= E\{V(X|P)\} + V\{E(X|P)\} \\ &= E\{nP(1-P)\} + V(nP) \\ &= n\,E(P) - n\,E(P^2) + n^2\,V(P) \\ &= \frac{n}{2} - n\left(\frac{1}{12} + \frac{1}{4}\right) + n^2 \frac{1}{12} \\ &= \frac{n^2 + 2n}{12} \end{aligned}$$

THANK YOU!