Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 6. Joint Distribution

6.3 Conditional Expected Values

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Conditional Expected Values

Let X and Y be discrete random variables, then the conditional expected value of $g(Y)$ given $X = x$ is

$$E\{g(Y)|X = x\} = \sum_{y} g(y) f_{Y|X=x}(y|X = x).$$

Let X and Y be continuous random variables, then the conditional expected value of $g(Y)$ given $X = x$ is

$$E\{g(Y)|X = x\} = \int_{-\infty}^{\infty} g(y) f_{Y|X=x}(y|X = x) dy.$$

Example 1

Let the discrete random variable X and Y have joint probability mass function $f(x, y)$ given by the entries in the table. Find the conditional expected value of Y given $X = 2$.

*	1	2	3	$f_X(x)$
1	0.2	0.1	0.3	0.6
2	0.1	0.1	0.2	0.4
$f_Y(y)$	0.3	0.2	0.5	1

$$E(\underbrace{Y}_{g(Y)} \mid X=2) = \sum_y y\, f_{Y|X=2}(y|X=2) = 1 * \frac{1}{4} + 2 * \frac{1}{4} + 3 * \frac{1}{2} = \frac{9}{4}$$

$$f_{Y|X=2}(y|X=2) = \frac{f(x,y)}{f_X(2)} = \begin{cases} 0.1/0.4 & y=1 \\ 0.1/0.4 & y=2 \\ 0.2/0.4 & y=3 \end{cases} = \begin{cases} 1/4 & y=1 \\ 1/4 & y=2 \\ 1/2 & y=3 \end{cases}$$

Theorem

Theorem 6.6

If X and Y are independent random variables, then $Cov(X, Y) = 0$.

Example 2

A thief is in a fiendish, dark,m circular dungeon with three identical doors. Once the thief chooses a door and passes through it, the door locks behind him. The three doors lead to the following three outcomes.

- Door 1 leads to a 6 hour tunnel that leads to freedom
- Door 2 leads to a 3 hour tunnel that returns him to the dungeon
- Door 3 leads to a 9 hour tunnel that returns him to the dungeon

Each door is chosen with equal probability. When the thief is dropped back into the dungeon after choosing the second or third door, he is a “Markov" thief in the sense that there is a memoryless subsequent choice of doors. He isn’t able to mark the doors in any way. What is his expected time to escape?

Let X = time to escape. Y = the door number chosen initially by the thief.

$$E(X) = \underbrace{E(X|Y=1)P(Y=1)} + \underbrace{E(X|Y=2)P(Y=2)} + \underbrace{E(X|Y=3)P(Y=3)}$$

$$= 6 * \frac{1}{3} + (3 + E(X)) * \frac{1}{3} + (9 + E(X)) * \frac{1}{3}$$

$$= 2 + 1 + 3 + \frac{2E(X)}{3}$$

$$\Rightarrow E(X) = 18$$

Example 3

Let the random variable X be the number of tails tossed prior to the first head in repeated tosses of a biased coin, where $p = P(\text{tossing a head})$ on an individual toss. Find $E(X)$.

$X \sim Geo(p)$. $\Rightarrow E(X)$?

Let Y = the result of 1st toss.

$f(x) = p(1-p)^x, \quad x = 0, 1, 2, \cdots$

$$E(X) = E(X \mid Y=0)P(Y=0) + E(X \mid Y=1)P(Y=1)$$

$$= (1 + E(X))(1-p) + 0 \cdot p$$

$$= 1 - p + E(X) - pE(X)$$

$$\Rightarrow E(X) = \frac{1-p}{p}$$

Example 4

For the continuous random variables X and Y with joint pdf

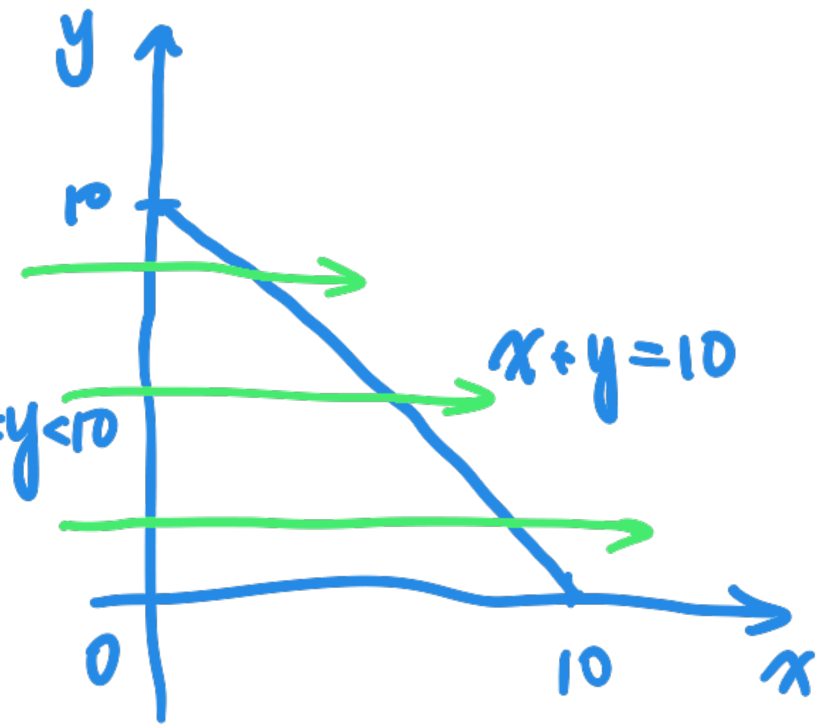
$$f(x,y) = \frac{1}{50} \quad x > 0,\ y > 0,\ x + y < 10,$$

find $f_{X|Y=y}(x|Y=y)$ and $E(X^k|Y=y)$ for some real constant $k > 0$.

$$f_{X|Y=y}(x|Y=y) = \frac{f(x,y)}{f_Y(y)} = \frac{1/50}{(10-y)/50} = \frac{1}{10-y}$$

$$f_Y(y) = \int_0^{10-y} f(x,y)\,dx = \int_0^{10-y} \frac{1}{50}\,dx = \frac{10-y}{50},\ 0<y<10$$

$$f_{X|Y=2}(x|Y=2) = \frac{1}{10-2} = \frac{1}{8},\quad 0<x<8$$



$$E\left(\underset{\sim}{X}^k \mid Y=y\right) = \int_0^{10-y} x^k \frac{1}{10-y}\,dx$$

$g(X)$

$$= \frac{1}{10-y}\left(\left.\frac{x^{k+1}}{k+1}\right|_0^{10-y}\right)$$

$$= \frac{(10-y)^k}{k+1}, \qquad 0 < y < 10$$

$Y=2.$ $$E\left(\underset{\sim}{X} \mid Y=2\right) = \frac{(10-y)^1}{1+1} = \frac{10-2}{2} = 4$$

THANK YOU!