Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 6. Joint Distribution

6.3 ~~Conditional Expected Values~~

Covariance

population mean μ.
population variance σ^2

population covariance.

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Covariance

$V(X) = E\{(X-\mu_X)^2\}$

$Cov(X,Y) = E\{(X-\mu_X)(Y-\mu_Y)\}$

Covariance

Let X and Y be random variables with finite population means μ_X and μ_Y, respectively. The population covariance between X and Y is

$$Cov(X, Y) = E\{(X - \mu_X)(Y - \mu_Y)\}.$$

$Cov(X,Y) = E\{(X-\mu_X)(Y-\mu_Y)\}$

$\parallel$

$Cov(Y,X) = E\{(Y-\mu_Y)(X-\mu_X)\}$

- Symmetric in its arguments: $Cov(X, Y) = Cov(Y, X)$
- Defining formula useful for conceptualizing covariance
  - If X and Y tend to be on opposite sides of their means together $\Rightarrow$ population covariance negative
  - If X and Y tend to be on the same sides of their means together $\Rightarrow$ population covariance positive

$Cov(X,Y)$, $C(X,Y)$, σ_{XY}

Example 1

A fair coin is tossed twice. Let X be the number of heads that appear and Y be the number of tails that appear. Find the population covariance between X and Y.

$$\mathrm{Cov}(X,Y) = E\{(X-\mu_X)(Y-\mu_Y)\}$$

$f(x,y)$

$$S = \{(x,y) \mid (x,y) = (0,2), (1,1), (2,0)\}$$

$$f(x,y) = \begin{cases} \frac{1}{2}\cdot\frac{1}{2} & x=0,\ y=2 \\ \frac{1}{2}\cdot\frac{1}{2}+\frac{1}{2}\cdot\frac{1}{2} & x=1,\ y=1 \\ \frac{1}{2}\cdot\frac{1}{2} & x=2,\ y=0 \end{cases} = \begin{cases} \frac{1}{4} & x=0,\ y=2 \\ \frac{1}{2} & x=1,\ y=1 \\ \frac{1}{4} & x=2,\ y=0 \end{cases}$$

$x \backslash y$	0	1	2
0	0	0	$\frac{1}{4}$
1	0	$\frac{1}{2}$	0
2	$\frac{1}{4}$	0	0

$$f_X(x) = \begin{cases} \frac{1}{4} & x=0 \\ \frac{1}{2} & x=1 \\ \frac{1}{4} & x=2 \end{cases} \qquad f_Y(y) = \begin{cases} \frac{1}{4} & y=0 \\ \frac{1}{2} & y=1 \\ \frac{1}{4} & y=2 \end{cases}$$

$$\mu_X = \sum x f_X(x) = 0\times\frac{1}{4} + 1\times\frac{1}{2} + 2\times\frac{1}{4} = 1$$

$$\mu_Y = \sum y f_Y(y) = 1$$

$$\mathrm{Cov}(X,Y) = E\{(X-\mu_X)(Y-\mu_Y)\}$$

$$= \sum (x-1)(y-1)f(x,y)$$

$$= (0-1)(2-1)\tfrac{1}{4} + (1-1)(1-1)\tfrac{1}{2} + (2-1)(0-1)\tfrac{1}{4}$$

$$= -\tfrac{1}{2}$$

$$Y = 2 - X$$

units of Cov: product of the units of X and unit of Y

X. Y. Z R.V.

pairwise Cov: Cov(X,Y), Cov(X,Z)
Cov(Y,Z)

Special Case

V(X) = Cov(X,X)

Var-Cov Matrix

Variance is a special case of covariance: $V(X) = Cov(X, X)$.

- **Bivariate Case:**

Cov(X,X) Cov(X,Y)

$$\Sigma = \begin{pmatrix} V(X) & Cov(X,Y) \\ Cov(Y,X) & V(Y) \end{pmatrix}$$

Cov(Y,X) Cov(Y,Y)

- **Trivariate Case:**

$$\Sigma = \begin{pmatrix} V(X) & Cov(X,Y) & Cov(X,Z) \\ Cov(Y,X) & V(Y) & Cov(Y,Z) \\ Cov(Z,X) & Cov(Z,Y) & V(Z) \end{pmatrix}$$

positive semi-definite

Theorem 6.4

If X and Y are random variables with finite population means μ_X and μ_Y, respectively, then

$$Cov(X, Y) = E\{(X - \mu_X)(Y - \mu_Y)\} = E(XY) - \mu_X\mu_Y.$$

$$E\{(X-\mu_X)(Y-\mu_Y)\} = E\{XY - \mu_X Y - \mu_Y X + \mu_X\mu_Y\}$$

$$= E(XY) - \mu_X E(Y) - \mu_Y E(X) + \mu_X\mu_Y$$

$$= E(XY) - \mu_X\mu_Y - \cancel{\mu_X\mu_Y} + \cancel{\mu_X}\mu_Y$$

$$= E(XY) - \mu_X\mu_Y$$

Example 2

A fair coin is tossed twice. Let X be the number of heads that appear and Y be the number of tails that appear. Find the population covariance between X and Y using the shortcut formula

$$Cov(X,Y) = \underbrace{E(XY)} - \mu_X \mu_Y$$

$$= \sum_{x}\sum_{y} xy f(x,y) - 1*1$$

$$= 0*2*\tfrac{1}{4} + \underbrace{1*1*\tfrac{1}{2}} + 2*0*\tfrac{1}{4} - 1$$

$$= -\tfrac{1}{2}$$

Example 3

Deal two cards from a well-shuffled deck. Let X be the number of aces dealt and Y be the number of face cards dealt. Using the shortcut formula, find the population covariance between X and Y.

$S = \{(x,y) \mid (0,0), (0,1), (0,2), (1,0), (1,1), (2,0)\}$

$x \backslash y$	0	1	2	$f_X(x)$
0	630/1326	432/1326	66/1326	1128/1326
1	144/1326	48/1326	0	192/1326
2	6/1326	0	0	6/1326
$f_Y(y)$	780/1326	480/1326	66/1326	

$\mu_X = \sum x f_X(x) = \frac{2}{13}$

$\mu_Y = \sum y f_Y(y) = \frac{6}{13}$

$f(0,0) = \frac{\binom{36}{2}}{\binom{52}{2}}$

$f(0,1) = \frac{\binom{12}{1}\binom{36}{1}}{\binom{52}{2}}$

$f(0,2) = \frac{\binom{12}{2}}{\binom{52}{2}}$

$f(1,0) = \frac{\binom{4}{1}\binom{36}{1}}{\binom{52}{2}}$

$f(1,1) = \frac{\binom{4}{1}\binom{12}{1}}{\binom{52}{2}}$

$f(2,0) = \frac{\binom{4}{2}}{\binom{52}{2}}$

THANK YOU!