Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 6. Joint Distribution

6.3 Expected Values

$E(X)$

$E\{g(X)\}$

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Expected Values

Let X and Y be random variables with joint probability mass function $f(x, y)$ if the random variables are discrete or joint probability density function $f(x, y)$ if the random variables are continuous. The expected value of $g(X, Y)$ is

$$E\{g(X,Y)\} = \begin{cases} \sum_x \sum_y g(x,y) f(x,y) & X, Y \text{ discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} g(x,y) f(x,y) dydx & X, Y \text{ continuous} \end{cases}$$

- The expected value of a function of two random variables $g(X, Y)$ is simply the sum or integral of the product of the functions $g(x, y)$ and $f(x, y)$.

Example 1

Find $E(X+Y)$ for the discrete random variables X and Y with joint pmf

$$f(x,y)=\begin{cases} 0.2 & x=1, y=1 \\ 0.1 & x=1, y=2 \\ 0.3 & x=1, y=3 \\ 0.1 & x=2, y=1 \\ 0.1 & x=2, y=2 \\ 0.2 & x=2, y=3 \end{cases}.$$

$$E(\underbrace{X+Y}_{g(X,Y)}) = \sum_{A}\sum_{B} g(x,y)\,f(x,y)$$

$$= (1+1)*0.2+(1+2)*0.1+(1+3)*0.3+(2+1)*0.1+(2+2)*0.1+(2+3)*0.2$$

$$= 0.4+0.3+1.2+0.3+0.4+1.0 = 3.6$$

Example 2

Find $E(X^2 Y)$ for X and Y with joint pdf

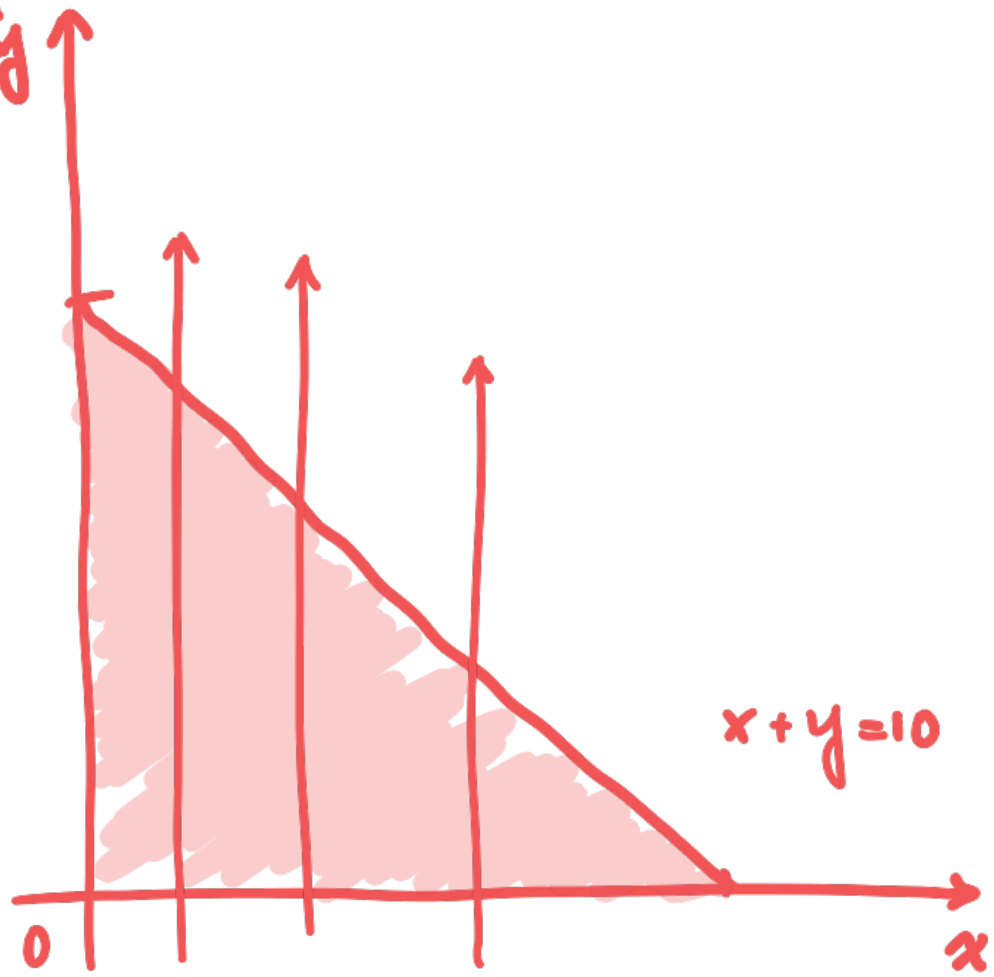
$$f(x, y) = \frac{1}{50} \quad x > 0, y > 0, x + y < 10.$$

$$E(\underbrace{X^2 Y}_{g(X,Y)}) = \int_0^{10} \int_0^{10-x} g(x,y) f(x,y)\, dy\, dx$$

$$= \int_0^{10} \int_0^{10-x} x^2 y \frac{1}{50}\, dy\, dx$$

$$= \int_0^{10} \frac{x^2 (10-x)^2}{100}\, dx$$

$$= \frac{100}{3}$$



Theorem 6.2

If X and Y are random variables, then

$$E\{g(X) + h(Y)\} = E\{g(X)\} + E\{h(Y)\}$$

for any functions g and h.

Proof: Let $f(x,y)$ denote the joint probability density fn of X & Y.

$$E\{g(X)+h(Y)\} = \int_{\mathbb{R}}\int_{\mathbb{R}} \{g(x)+h(y)\} f(x,y)\,dy\,dx$$

$$= \iint_{\mathbb{R}} \{g(x)f(x,y) + h(y)f(x,y)\}\,dy\,dx$$

$$= \iint_{\mathbb{R}} g(x)f(x,y)\,dy\,dx + \iint_{\mathbb{R}} h(y)f(x,y)\,dy\,dx$$

$$= E\{g(X)\} + E\{h(Y)\}$$

Theorem 6.3

If X and Y are independent random variables, then

$$E\{g(X)h(Y)\} = E\{g(X)\}E\{h(Y)\}$$

for any functions g and h.

Proof. X & Y are independent $\Rightarrow f(x,y) = f_X(x) f_Y(y)$

$$E\{g(X)h(Y)\} = \iint_{\mathcal{A}} g(x)h(y)f(x,y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x)\,h(y)\,f_X(x)\,f_Y(y)\,dy\,dx$$

$$= \int_{-\infty}^{\infty} g(x) f_X(x)\,dx * \int_{-\infty}^{\infty} h(y) f_Y(y)\,dy$$

$$= E(g(X)) * E\{h(Y)\}$$

THANK YOU!