Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 6. Joint Distribution

6.2 Independent Random Variables

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Theorem

Theorem 6.1

Let the random variables X and Y have joint distributioon described by $f(x, y)$ defined on a product space. Then X and Y are independent iff $f(x, y)$ can be written as the product of a function of x only and a function of y only.

Denote the non-negative fn of X as $g_1(x)$
the non-negative fn of Y as $g_2(y)$
the support of X is $\mathcal{A}$, support of Y is B.

1st. X & Y are indep $\Rightarrow f(x,y) = g_1(x)\, g_2(y)$

$\Rightarrow f(x,y) = \underbrace{f_X(x)}_{g_1(x)} * \underbrace{f_Y(y)}_{g_2(y)}$, $x \in \mathcal{A}$, $y \in B$.

2nd. $f(x,y) = g_1(x) * g_2(y) \Rightarrow X$ & Y are indep.

$f_X(x) = \int_B f(x,y)\,dy = \int_B g_1(x)\, g_2(y)\,dy = g_1(x) \int_B g_2(y)\,dy = C_1\, g_1(x)$

$f_Y(y) = \int_{\mathcal{A}} f(x,y)\,dx = \int_{\mathcal{A}} g_1(x)\, g_2(y)\,dx = g_2(y) \int_{\mathcal{A}} g_1(x)\,dx = C_2\, g_2(y)$

$$f_X(x) = c_1 g_1(x) \Rightarrow \int_{\mathcal{A}} f_X(x)\,dx = \int_{\mathcal{A}} c_1 g_1(x)\,dx = 1 = c_1 \int_{\mathcal{A}} g_1(x)\,dx = 1$$

$$f_Y(y) = c_2 g_2(y) \Rightarrow \int_B f_Y(y)\,dy = \int_B c_2 g_2(y)\,dy = c_2 \int_B g_2(y)\,dy = 1$$

$$\int_{\mathcal{A}} \int_B f(x,y)\,dy\,dx = \int_{\mathcal{A}} \int_B g_1(x)\, g_2(y)\,dy\,dx$$

$$= \int_{\mathcal{A}} g_1(x)\,dx \int_B g_2(y)\,dy =$$

$$= \frac{1}{c_1} \frac{1}{c_2} = 1 \Rightarrow c_1 c_2 = 1$$

$$f(x,y) = g_1(x)\, g_2(y) = c_1 c_2\, g_1(x)\, g_2(y)$$

$$= c_1 g_1(x)\; c_2 g_2(y) = f_X(x)\, f_Y(y)$$

$\Rightarrow$ X & Y are indep.

Example 2

Let X_1 and X_2 be random variables with joint pdf

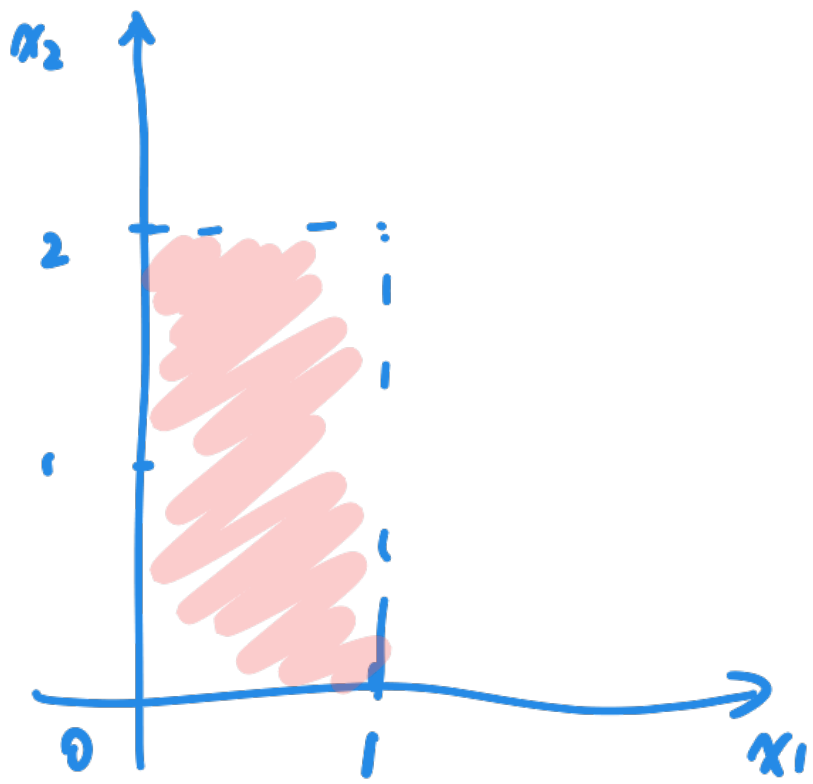
$$f(x_1, x_2) = x_1 x_2, \quad \underbrace{0 < x_1 < 1}_{A}, \ \underbrace{0 < x_2 < 2}_{B}$$

Are X_1 and X_2 independent?

①. product space.

②. $f(x_1, x_2) = \underbrace{x_1}_{g_1(x_1)} * \underbrace{x_2}_{g_2(x_2)}$

$\therefore$ X & Y are indep



Example 3

Are the random variables X and Y with jooint probability density function

$$f(x,y) = \frac{1}{50}, \quad x > 0,\ y > 0,\ x + y < 10$$

independent?

① product space? X

∴ X & Y are dependent

② $f(x,y) = g_1(x) \times g_2(y)$?

Example 4

Are the random variables X and Y with jooint probability density function

$$f(x, y) = xy - 2x - y + 2, \quad 0 < x < 1, \; 0 < y < 2$$

independent?

① . product space . ? ✓ $A = \{0 < x < 1\}$

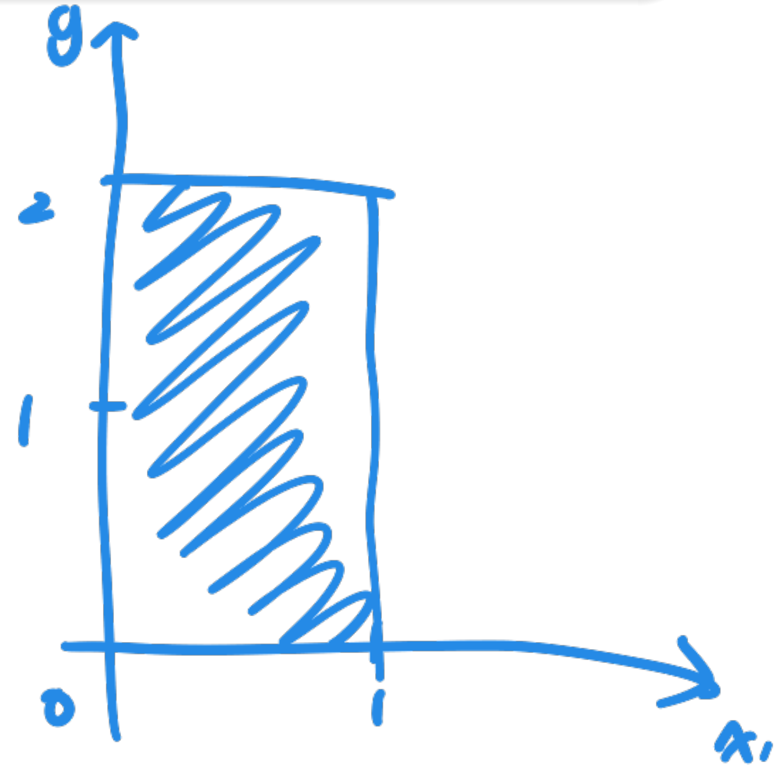
$B = \{0 < y < 2\}$

② $f(x, y) = g_1(x) * g_2(y)$?

$= (x-1) * (y-2)$ ✓

$xy - y - 2x + 2$ ✓

$\therefore$ X & Y are independent



Example 5

Are the random variables X and Y with jooint probability density function

$$f(x,y) = x + y, \quad 0 < x < 1,\ 0 < y < 1$$

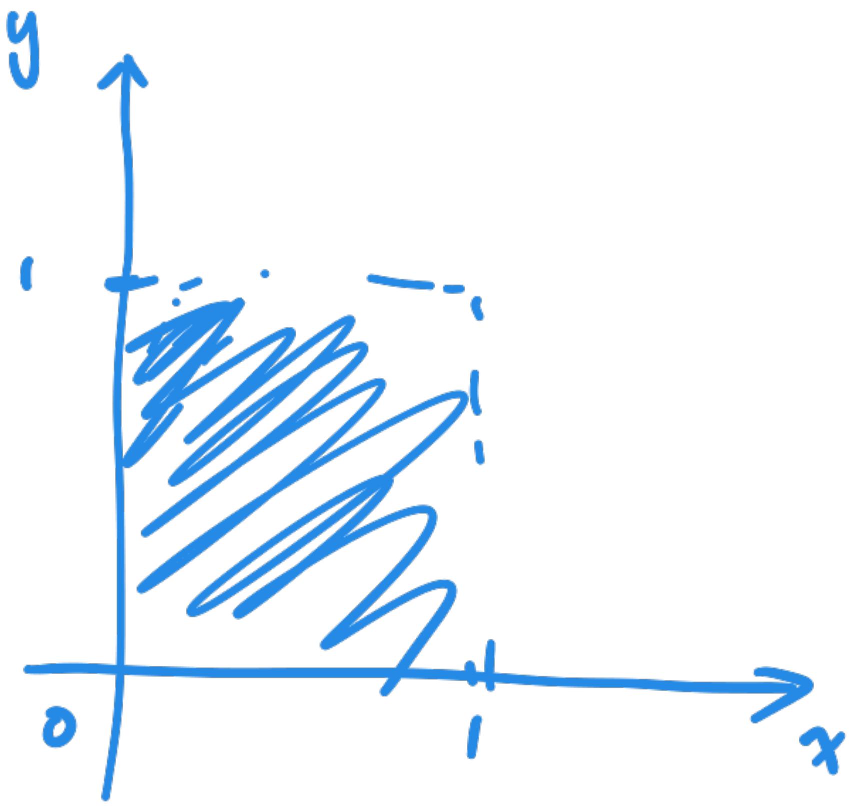
independent?

① product space? ✓ $A = \{0 < x < 1\}$

$B = \{0 < y < 1\}$

② $f(x,y) = g_1(x) * g_2(y)$? ✗

$\therefore$ X & Y are dependent



Example 6

Let $X_1 \sim \mathrm{Exp}(\lambda_1)$ and $X_2 \sim \mathrm{Exp}(\lambda_2)$ be independent random variables. Find the probability that $X_1 < X_2$

Joint Distribution.

X_1 & X_2 are independent $\Rightarrow$ $f(x_1, x_2) = f_{X_1}(x_1) * f_{X_2}(x_2)$

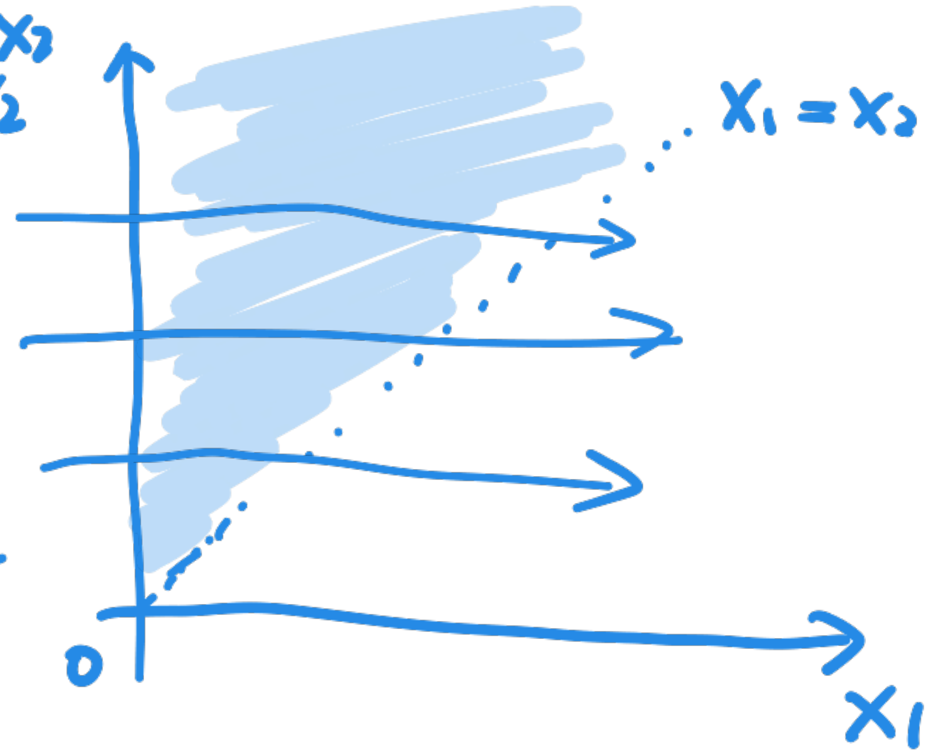
$$= \lambda_1 e^{-\lambda_1 x_1} \cdot \lambda_2 e^{-\lambda_2 x_2}, \quad 0 < x_1, x_2$$

$$P(X_1 < X_2) = \int_0^\infty \int_0^{x_2} \lambda_1 \lambda_2 e^{-\lambda_1 x_1} e^{-\lambda_2 x_2} \, dx_1 \, dx_2$$

$$= \int_0^\infty \lambda_2 e^{-\lambda_2 x_2} \left(-e^{-\lambda_1 x_1} \Big|_0^{x_2} \right) dx_2$$

$$= \int_0^\infty \lambda_2 e^{-\lambda_2 x_2} \left(1 - e^{-\lambda_1 x_2} \right) dx_2$$

$$= \frac{\lambda_1}{\lambda_1 + \lambda_2}$$



THANK YOU!