Department of Mathematics
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MATH 451/551

Chapter 6. Joint Distribution

6.2 Independent Random Variables

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events A & B. are independent iff. $P(A \cap B) = P(A)P(B)$

R.Vs X & Y. Let event $A = \{a_1 < X < a_2\}$

event $B = \{b_1 < Y < b_2\}$

A & B independent. $P(a_1 < X < a_2, b_1 < Y < b_2) = P(a_1 < X < a_2) * P(b_1 < Y < b_2)$

$f(x, y) = f_X(x) * f_Y(y)$

A & B. iff. $P(A|B) = P(A)$

$$P(B|A) = P(B)$$

Assume R.V. X & Y are continuous. $f(x,y)$

$$f_{X|Y=y}(x|Y=y) \quad \& \quad f_{Y|X=x}(y|X=x)$$

$f_{Y|X=x}(y|X=x)$ does NOT depend on value of x.

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx = \int_{-\infty}^{\infty} f_{Y|X=x}(y|X=x)\, f_X(x)\,dx$$

$$= f_{Y|X=x}(y|X=x) \underbrace{\int_{-\infty}^{\infty} f_X(x)\,dx}_{1}$$

$$= f_{Y|X=x}(y|X=x)$$

Independent Random Variables

Let the random variables X and Y (discrete or continuous) have a joint distribution described by $f(x, y)$ and marginal distributions described by $f_X(x)$ and $f_Y(y)$. The random variables X and Y are independent iff $\mathbf{f(x, y) = f_X(x) f_Y(y)}$ for all real numbers x and y.

- Intuitively, if the value of X does not affect the distribution of Y and if the value of Y does not affect the distribution of X, then X and Y are **independent**.
- Random variables that are not independent are **dependent**.
- An equivalent definitioon can also be written in terms of cumulative distribution: the random variables X and Y are independent iff $\mathbf{F(x, y) = F_X(x) F_Y(y)}$ for all real numbers x and y.
- For X and Y to be independent, the support of their joint distribution must be a **product space**, i.e., if X has support $\mathcal{A}$ and Y has support $\mathcal{B}$, then the product space is $\{(\mathbf{x}, \mathbf{y}) | \mathbf{x} \in \mathcal{A} \text{ and } \mathbf{y} \in \mathcal{B}\}$

Example 1

Are the random variables X and Y with joint pmf given below independent?

	1	2	3	$f_X(x)$
1	0.2	0.1	0.3	0.6
2	0.1	0.1	0.2	0.4
$f_Y(y)$	0.3	0.2	0.5	

$f(x,y) \overset{?}{=} f_X(x) f_Y(y)$

$f(1,1) \overset{?}{=} f_X(1) * f_Y(1)$

$0.2 \neq 0.6 * 0.3 = 0.18$

$\therefore$ X & Y are NOT independent

Example 2

Let X_1 and X_2 be random variables with joint pdf

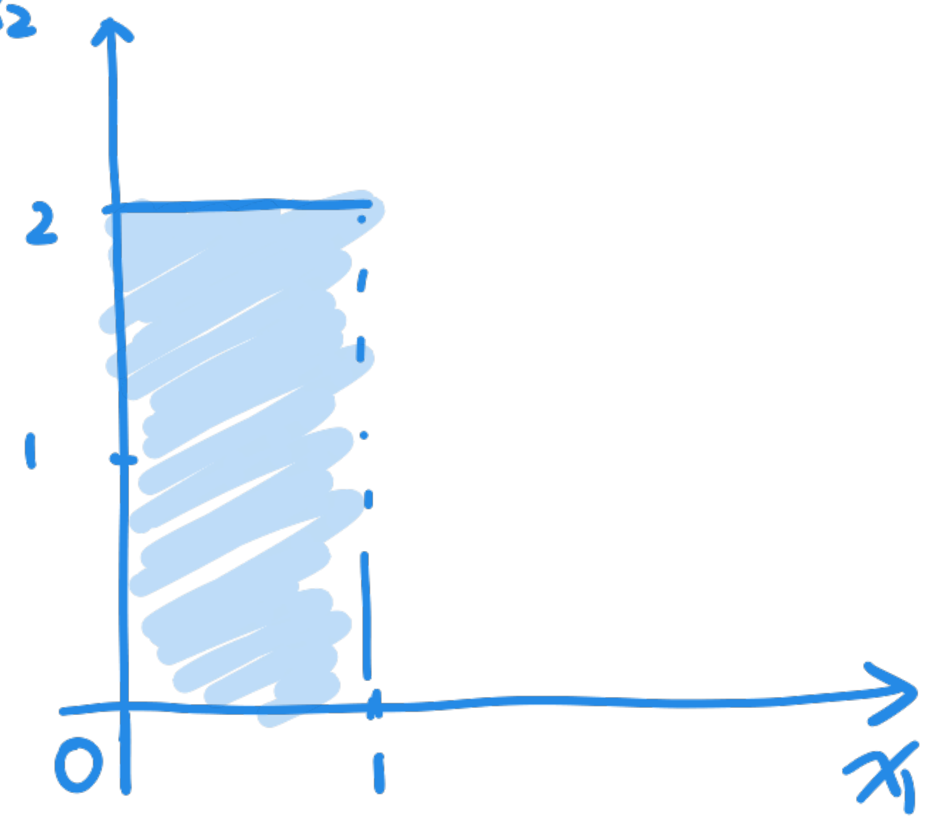
$$f(x_1, x_2) = x_1 x_2, \quad \underbrace{0 < x_1 < 1,\ 0 < x_2 < 2}_{\text{product space } \checkmark}$$

Are X_1 and X_2 independent?

$$f_{X_1}(x_1) = \int_0^2 f(x_1, x_2)\,dx_2 = \int_0^2 x_1 x_2\,dx_2 = \left.\frac{x_1 x_2^2}{2}\right|_0^2 {}_{x_2}$$

$$= 2x_1, \quad 0 < x_1 < 1$$

$f_{X_2}(x_2)$



THANK YOU!