Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 6. Joint Distribution

6.1 Bivariate Distribution

A conditional distribution is the distribution of one random variable given the value of another random variable.

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$$P(A \mid B) = \frac{P(A \cap B)}{P(B)}$$

$$f_X(x) = P(X=x)$$

Discrete R.V: X & Y.

pmf: $\underbrace{f_{\underset{\uparrow \text{R.V}}{X} \mid \underset{\uparrow \text{R.V}}{Y} = y}(x \mid Y=y)} = P(\overbrace{\underset{\uparrow \text{R.V}}{X} = \underset{\uparrow \text{value}}{x}}^{A} \mid \overbrace{\underset{\uparrow \text{R.V}}{Y} = \underset{\uparrow \text{value}}{y}}^{B})$

$$= \frac{P(X=x,\ Y=y)}{P(Y=y)}$$

$$= \frac{f(x,y)}{f_Y(y)}$$

$$f_{Y|X=x}(y \mid X=x) = P(Y=y \mid X=x) = \frac{f(x,y)}{f_X(x)}$$

Continuous R.V.: X & Y.

$$P(\underset{\sim}{y} < Y < y + \underset{\sim}{dy}) \cong \underline{f_Y(y)\,dy}.$$

pdf: $f_{X|Y=y}(x|Y=y) \equiv P(\overbrace{x < X < x+dx}^{A} \mid \overbrace{y < Y < y+dy}^{B})$

$\underset{\text{R.V}}{\uparrow}$ $\underset{\text{R.V}}{\uparrow}$ $\underset{\text{value}}{\uparrow}$

$$= \frac{P(x < X < x+dx,\ y < Y < y+dy)}{P(y < Y < y+dy)}$$

$$= \frac{f(x,y)\,\underline{dx}\,\cancel{dy}}{f_Y(y)\,\cancel{dy}} = \frac{f(x,y)}{f_Y(y)}\,dx.$$

Conditional Distribution

Conditional Distribution

Let the (discrete or continuous) random variables X and Y have a joint distribution described by $f(x, y)$ and marginal distributions described by $f_X(x)$ and $f_Y(y)$. The conditional distribution of X given $Y = y$ is

$$f_{X|Y}(x|Y = y) = \frac{f(x, y)}{f_Y(y)}$$

for $f_Y(y) > 0$, defined over the appropriate support. Likewise, the conditional distribution of Y given $X = x$ is

$$f_{Y|X}(y|X = x) = \frac{f(x, y)}{f_X(x)}$$

for $f_X(x) > 0$, defined over the appropriate support.

- These conditional probability mass functions and probability density functions satisfy the standard existence conditions.

proof: Discrete R.V.

$$\sum_x f_{X|Y=y}(x|Y=y) = \sum_x \frac{f(x,y)}{f_Y(y)} = \frac{1}{f_Y(y)} \sum_x f(x,y) = \frac{1}{f_Y(y)} f_Y(y) = 1$$

Continuous R.V.

$$\int_{-\infty}^{\infty} f_{X|Y=y}(x|Y=y)\,dx = \int_{-\infty}^{\infty} \frac{f(x,y)}{\boxed{f_Y(y)}}\,dx = \frac{1}{f_Y(y)} \int_{-\infty}^{\infty} f(x,y)\,dx$$

$$= \frac{1}{f_Y(y)} f_Y(y) = 1$$

Example 1

Given following joint distribution

$$f(x,y) = \begin{cases} 0.2 & x = 1, y = 1 \\ 0.1 & x = 1, y = 2 \\ 0.3 & x = 1, y = 3 \\ 0.1 & x = 2, y = 1 \\ 0.1 & x = 2, y = 2 \\ 0.2 & x = 2, y = 3 \end{cases},$$

X \ Y	1	2	3	$f_X(x)$
1	0.2	0.1	0.3	0.6
2	0.1	0.1	0.2	0.4
$f_Y(y)$	0.3	0.2	0.5	1

find the probability mass functions associated with the following conditional random variables:

$$f_{X|Y}(x|Y=1) \quad \text{and} \quad f_{Y|X}(y|X=2)$$

$$f_{X|Y=1}(x|Y=1) = \frac{f(x,y)}{f_Y(1)} = \begin{cases} \frac{0.2}{0.3} & x=1 \\ \frac{0.1}{0.3} & x=2 \end{cases} = \begin{cases} 2/3 & x=1 \\ 1/3 & x=2 \end{cases}$$

$$f_{Y|X=2}(y|X=2) = \frac{f(x,y)}{f_X(2)} = \begin{cases} 0.1/0.4 & y=1 \\ 0.1/0.4 & y=2 \\ 0.2/0.4 & y=3 \end{cases} = \begin{cases} 1/4 & y=1 \\ 1/4 & y=2 \\ 1/2 & y=3 \end{cases}$$

36 Other cards
4 Aces
12 Face cards

Example 2

Deal two cards from a well-shuffled deck. Let the random variable X be the number of aces dealt and let the random variable Y be the number of face cards dealt. Find the conditional distribution of the number of aces in the hand given that there is one face card in the hand. $f_{X|Y=1}(x|Y=1)$

$x \backslash y$	0	1	2	f_X
0	$\binom{36}{2}/\binom{52}{2}$	$\binom{12}{1}\binom{36}{1}/\binom{52}{2}$	$\binom{12}{2}/\binom{52}{2}$	$1128/1326$
1	$\binom{4}{1}\binom{36}{1}/\binom{52}{2}$	$\binom{4}{1}\binom{12}{1}/\binom{52}{2}$	0	$192/1326$
2	$\binom{4}{2}/\binom{52}{2}$	0	0	$6/1326$
$f_Y(y)$	$780/1326$	$480/1326$	$66/1326$	1

$$f_{X|Y=1}(x|Y=1) = \begin{cases} f(0,1)/f_Y(1) & x=0 \\ f(1,1)/f_Y(1) & x=1 \end{cases}$$

$$= \begin{cases} 432/480 & x=0 \\ 48/480 & x=1 \end{cases}$$

$$= \begin{cases} 9/10 & x=0 \\ 1/10 & x=1 \end{cases}$$

Example 3

Let X and Y have joint pdf

$$f(x,y)=\frac{1}{50},\quad x>0, y>0, x+y<10$$

1. Find $f_{X|Y}(x|Y=y)$.
2. Find $P(3<X<5|Y=2)$.

$$f_Y(y)=\int_0^{10-y} f(x,y)\,dx=\int_0^{10-y}\frac{1}{50}\,dx$$

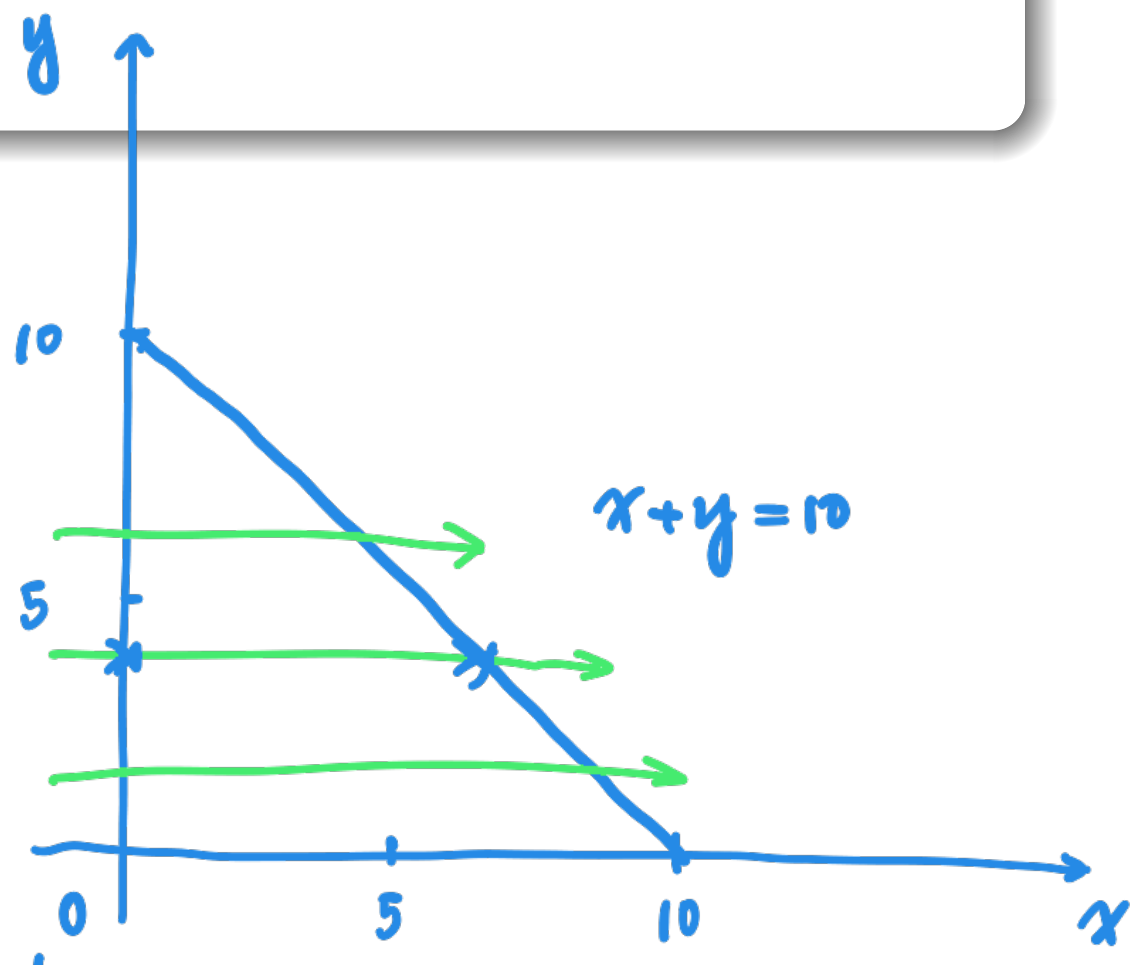
$$=\frac{10-y}{50},\quad 0<y<10$$

① $$f_{X|Y=y}(x|Y=y)=\frac{f(x,y)}{f_Y(y)}=\frac{\frac{1}{50}}{\frac{10-y}{50}}$$

$$=\frac{1}{10-y},\quad 0<x<10-y,\quad 0<y<10$$

② $$P(3<X<5|Y=2)=\int_3^5 f_{X|Y=2}\,dx$$

$$=\int_3^5\frac{1}{10-2}\,dx=\frac{5-3}{8}=\frac{1}{4}$$



THANK YOU!