Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 6. Joint Distribution

6.1 Bivariate Distribution

GuanNan Wang
gwang01@wm.edu

Joint Cumulative Distribution Functions

Joint CDF

The joint cumulative distribution function of two random variables (discrete or continuous) X_1 and X_2 is

① $0 \le F_X(x) \le 1$
② nondecreasing
③ $\lim_{x \to -\infty} F_X(x) = 0$
④ $\lim_{x \to \infty} F(x) = 1$

- $$F_{X_1,X_2}(x_1,x_2) = F(x_1, x_2) = P(X_1 \le x_1, X_2 \le x_2).$$

$$\lim_{x_1 \to -\infty} \lim_{x_2 \to -\infty} F_{X_1,X_2}(x_1,x_2) = 0, \quad \lim_{x_1 \to \infty} \lim_{x_2 \to \infty} F_{X_1,X_2}(x_1,x_2) = 1$$

- **Discrete Random Variables:** $F_X(x) = \sum_{w \le x} f_X(x)$

$$F(x_1, x_2) = P(X_1 \le x_1, X_2 \le x_2) = \sum_{w_1 \le x_1} \sum_{w_2 \le x_2} f(w_1, w_2)$$

- **Continuous Random Variables:** $F_X(x) = \int_{-\infty}^{x} f_X(w)\,dw$

$$F(x_1, x_2) = P(X_1 \le x_1, X_2 \le x_2) = \int_{-\infty}^{x_1} \int_{-\infty}^{x_2} f(w_1, w_2)\,dw_1\,dw_2,$$

and

$$f(x_1, x_2) = \frac{\partial^2 F(x_1, x_2)}{\partial x_1 \partial x_2}.$$

IV. $F_{X_1X_2}(x_1,x_2) = \int_0^1 \int_0^{x_2} x_1 x_2 \, dx_2 \, dx_1$

$= \int_0^1 \frac{x_1 x_2^2}{2} dx_1 = \frac{x_2^2}{4} \quad (x_1 \geq 1,\ 0 < x_2 < 2)$

Example 1

Let X_1 and X_2 be continuous random variables with joint probability density function

Continuous.

$$f(x_1, x_2) = x_1 x_2, \quad 0 < x_1 < 1,\ 0 < x_2 < 2.$$

Find the joint cumulative distribution function.

I. $F_{X_1,X_2}(x_1,x_2) = 0$; V. $F_{X_1,X_2}(x_1,x_2) = 1$

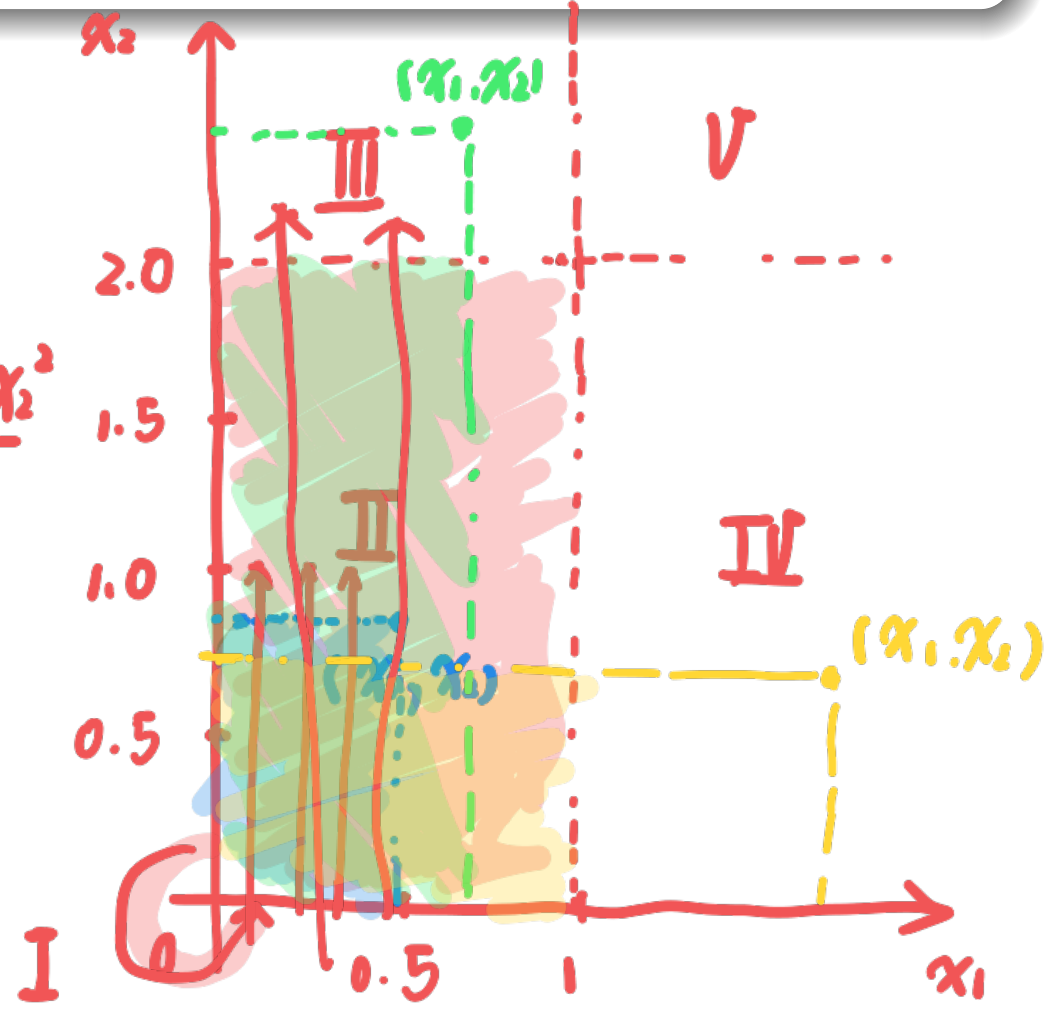
II. $F_{X_1,X_2}(x_1,x_2) = \int_0^{x_1} \int_0^{x_2} f(x_1,x_2) \, dx_2 \, dx_1$

$= \int_0^{x_1} \int_0^{x_2} x_1 \cdot x_2 \, dx_2 \, dx_1 = \int_0^{x_1} \frac{x_1 x_2^2}{2} dx_1 = \frac{x_1^2 x_2^2}{4}$

$(0 < x_1 < 1,\ 0 < x_2 < 2)$

III. $F_{X_1,X_2}(x_1,x_2) = \int_0^{x_1} \int_0^{2} x_1 x_2 \, dx_2 \, dx_1$

$= \int_0^{x_1} 2x_1 \, dx_1 = x_1^2 \quad (0 < x_1 < 1,\ x_2 \geq 2)$



Example 2

A bag contains 6 red balls, 7 white balls, and 8 blue balls. A random sample of 5 balls is drawn without replacement from the bag. If X_1 denotes the number ~~oo~~of red balls in the sample and X_2 denotes the number of the white balls in the sample, what is $F(2,3)$?

$F(X_1=2, X_2=3)$
$= P(X_1 \le 2, X_2 \le 3)$

$$\mathcal{A} = \left\{ (x_1, x_2) \;\middle|\; \begin{array}{l} (0,0), (0,1), (0,2), (0,3), (0,4), (0,5), \\ (1,0), (1,1), (1,2), (1,3), (1,4), \\ (2,0), (2,1), (2,2), (2,3), \\ (3,0), (3,1), (3,2), (4,0), (4,1), (5,0) \end{array} \right\}$$

Discrete

$$= \{ (x_1, x_2) \mid x_1 = 0, 1, \cdots, 5,\ x_2 = 0, 1, \cdots, 5,\ x_1 + x_2 \le 5 \}$$

$$f_{X_1, X_2}(x_1, x_2) = \frac{\binom{6}{x_1}\binom{7}{x_2}\binom{8}{5-x_1-x_2}}{\binom{21}{5}}$$

$$F(2,3) = P(X_1 \le 2, X_2 \le 3)$$

$$= f(0,0) + f(0,1) + f(0,2) + f(0,3) + f(1,0) + f(1,1) + f(1,2) + f(1,3) + f(2,0) + f(2,1) + f(2,2)$$

$+ f(2,3) = 0.8603$

Univariate Cumulative Distribution

The univariate cumulative distribution function of one of the variables can be obtained by allowing the argument for the other variable to approach infinity.

III. IV.

$$
\begin{aligned}
F_X(x) &= P(X \le x) = P(X \le x, Y < \infty) \\
&= \lim_{y \to \infty} P(X \le x, Y \le y) \\
&= \lim_{y \to \infty} F(x, y) \\
&= F(x, \infty)
\end{aligned}
$$

Thank You

THANK YOU!