Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 5. Common Continuous Distribution

5.3 Gamma Distribution

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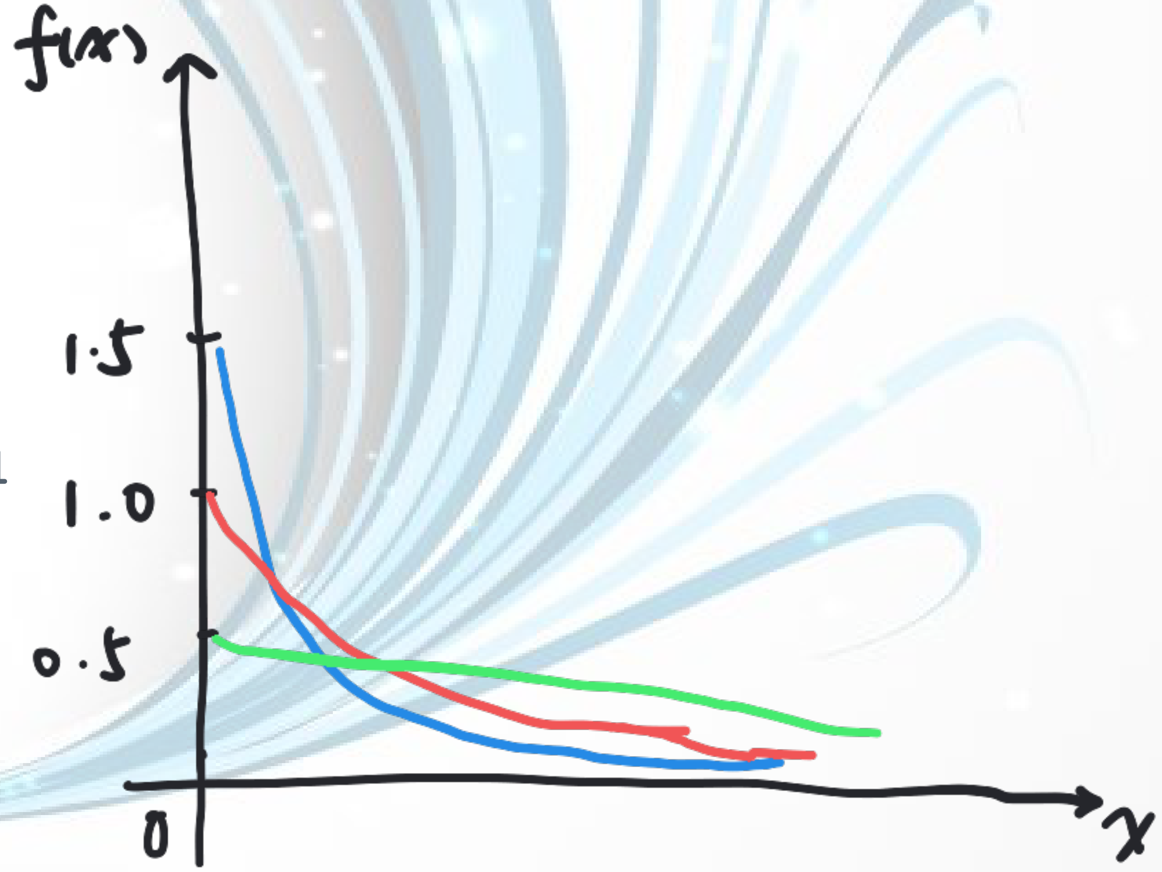

$$\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} \, dx \, , \qquad k > 0$$

$$y = \frac{x}{\lambda} \, , \quad \lambda > 0 \Rightarrow x = \lambda \cdot y \qquad dx = \lambda \, dy .$$

$$\Gamma(k) = \int_0^\infty (\lambda y)^{k-1} e^{-\lambda y} \lambda \, dy$$

$$1 = \int_0^\infty \frac{\lambda^k y^{k-1} e^{-\lambda y}}{\Gamma(k)} \, dy$$

Gamma Distribution

A continuous random variable X with pdf

$X \sim Gamma(\lambda, \kappa)$

$$f(x) = \frac{\lambda^\kappa y^{\kappa-1} e^{-\lambda x}}{\Gamma(\kappa)}, \quad x > 0$$

for some real constants $\lambda > 0$ and $\kappa > 0$ is a *gamma*(λ, κ) random variable.

Mean

$$E(X) = \int_0^\infty x f(x)\,dx = \int_0^\infty x \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}\,dx$$

$$= \int_0^\infty \frac{\lambda^k x^k e^{\boxed{-\lambda x}}}{\Gamma(k)}\,dx$$

Let $y = \lambda x$

then $x = \frac{1}{\lambda} y \quad dx = \frac{1}{\lambda} dy$

$$\underline{\underline{y=\lambda x}} \int_0^\infty \frac{\cancel{\lambda^k} \left(\frac{y}{\cancel{\lambda}}\right)^k e^{-y} \frac{1}{\lambda}}{\Gamma(k)}\,dy$$

$$= \int_0^\infty \frac{y^k e^{-y}}{\boxed{\lambda \Gamma(k)}}\,dy = \frac{1}{\lambda \Gamma(k)} \int_0^\infty y^k e^{-y}\,dy$$

$$= \frac{\Gamma(k+1)}{\lambda \Gamma(k)} = \frac{k \Gamma(k)}{\lambda \Gamma(k)} = \frac{k}{\lambda}$$

Variance

$$V(X) = E(X^2) - \{E(X)\}^2$$

$$E(\underbrace{X^2}_{g(X)}) = \int_0^\infty x^2 \frac{\lambda^k x^{k-1} e^{-\lambda x}}{\Gamma(k)}\, dx = \int_0^\infty \frac{\lambda^k x^{k+1} e^{-\lambda x}}{\Gamma(k)}\, dx$$

$$\underset{=}{y=\lambda x}\ \int_0^\infty \frac{\cancel{\lambda^k} \left(\frac{y}{\lambda}\right)^{k+1} e^{-y}}{\Gamma(k)} \frac{1}{\lambda}\, dy$$

Let $y = \lambda x$

then $x = \frac{1}{\lambda} y$

$dx = \frac{1}{\lambda} dy$

$$= \int_0^\infty y^{k+1} e^{-y} \boxed{\frac{1}{\lambda^2 \Gamma(k)}}\, dy$$

$$= \frac{\Gamma(k+2)}{\lambda^2 \Gamma(k)} = \frac{(k+1)\Gamma(k+1)}{\lambda^2 \Gamma(k)} = \frac{(k+1)k\cancel{\Gamma(k)}}{\lambda^2 \cancel{\Gamma(k)}} = \frac{k(k+1)}{\lambda^2}$$

$$V(X) = \frac{k^2+k}{\lambda^2} - \frac{k^2}{\lambda^2} = \frac{k}{\lambda^2}$$

MGF

$$M_X(t) = E(\underbrace{e^{tX}}_{g(X)}) = \int_0^\infty \underbrace{e^{tx}}\, \frac{\lambda^k x^{k-1} \underbrace{e^{-\lambda x}}}{\Gamma(k)}\, dx$$

$$= \int_0^\infty \frac{\lambda^k x^{k-1} e^{\boxed{-(\lambda-t)x}}}{\Gamma(k)}\, dx$$

Let $y = (\lambda - t)x$

then $x = \frac{1}{\lambda - t}\, y$

$dx = \frac{1}{\lambda-t}\, dy$

$$\underline{\underline{y=(\lambda-t)x}} \quad \int_0^\infty \frac{\lambda^k \left(\frac{y}{\lambda-t}\right)^{k-1} e^{-y}\, \frac{1}{\lambda-t}}{\Gamma(k)}\, dy$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^k \cancel{\frac{1}{\Gamma(k)} \int_0^\infty y^{k-1} e^{-y}\, dy}$$

$$= \left(\frac{\lambda}{\lambda-t}\right)^k, \quad t < \lambda$$

R Functions

Function	Returned Value
dgamma(x, κ, λ)	calculates the probability density function $f(x)$
pgamma(x, κ, λ)	calculates the cumulative distribution function $F(x)$
qgamma(u, κ, λ)	calculates the percentile (quantile) $F^{-1}(u)$
rgamma(m, κ, λ)	generates m random variates

Special Cases

- **Special Case 1:** Exponential distribution when $\kappa = 1$

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0.$$

$$E(X) = \frac{1}{\lambda}, \quad V(X) = \frac{1}{\lambda^2}, \quad E\left\{\left(\frac{X-\mu}{\sigma}\right)^3\right\} = 2, \quad E\left\{\left(\frac{X-\mu}{\sigma}\right)^4\right\} = 9.$$

- **Special Case 2:** Erlang distribution when κ is a positive integer k

$$f(x) = \frac{\lambda^k x^{k-1} e^{-\lambda x}}{(k-1)!}, \quad x > 0$$

$\Gamma(k)$

$$E(X) = \frac{k}{\lambda}, \; V(X) = \frac{k}{\lambda^2}, \; E\left\{\left(\frac{X-\mu}{\sigma}\right)^3\right\} = \frac{2}{\sqrt{k}},$$

$$E\left\{\left(\frac{X-\mu}{\sigma}\right)^4\right\} = \frac{3(k+2)}{k}.$$

Special Cases

- **Special Case 3:** χ^2 distribution when κ is $k/2$, where k is a parameter known as the degrees of freedom, and $\lambda = 1/2$,

$$f(x) = \frac{x^{k/2-1}e^{-x/2}}{\Gamma(k/2)2^{k/2}}, \quad x > 0$$

$$E(X) = k, \; V(X) = 2k, \; E\left\{\left(\frac{X-\mu}{\sigma}\right)^3\right\} = \frac{2\sqrt{2}}{\sqrt{k}},$$

$$E\left\{\left(\frac{X-\mu}{\sigma}\right)^4\right\} = \frac{3(k+4)}{k}.$$

Example 1

Suppose $X_1, X_2, \cdots, X_k$ are independent random variables with pdf $f(x) = \lambda e^{-\lambda x}$, $\lambda > 0$, $x > 0$, let $Y = \sum_{i=1}^{K} X_i$, what is the distribution of Y?

$X_i \overset{iid}{\sim} \mathcal{E}xp(\lambda) \quad i=1,\cdots,k$

$M_{X_i}(t) = \frac{\lambda}{\lambda - t}, \quad t<\lambda$

$M_Y(t) = E(e^{tY}) = E(e^{t(X_1+X_2+\cdots+X_k)})$

$= E(e^{tX_1} \cdot e^{tX_2} \cdot \ldots \cdot e^{tX_k})$

$\underline{\text{indep}}\ E(e^{tX_1})\, E(e^{tX_2}) \cdots E(e^{tX_k})$

$= M_{X_1}(t)\, M_{X_2}(t) \cdots M_{X_k}(t)$

$\underline{\text{identical}}\ \{M_{X_i}(t)\}^k = \left(\frac{\lambda}{\lambda-t}\right)^k, \quad t<\lambda$

$\Rightarrow Y \sim \text{Gamma}(\lambda, k)$

k is a positive integer

$\Rightarrow Y \sim \text{Erlang}(\lambda, k)$

Summary

- Gamma distribution is a generalization of the Exponential distribution
- Probability density function $Y \sim Gamma(\alpha, k)$

$$f(y) = \frac{\lambda^{\kappa} y^{\kappa-1} e^{-\lambda y}}{\Gamma(\kappa)}, \quad y > 0$$

- Positive scale parameter λ
- Positive shape parameter κ
- Population mean and variance

$$\mu = \frac{\kappa}{\lambda} \quad \text{and} \quad \sigma^2 = \frac{\kappa}{\lambda^2}$$

- Important special cases
  - Exponential distribution when $\kappa = 1$
  - Erlang distribution when κ is a positive integer k
  - χ^2 distribution when $\lambda = 1/2$ and $\kappa = k/2$

THANK YOU!