Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 5. Common Continuous Distribution

5.2 Exponential Distribution

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Exponential Distribution

Exponential Distribution

- A continuous random variable with positive support $\mathcal{A} = \{x > 0\}$ is useful in a variety of applications. Examples include
  - patient survival time after the diagosis of a particular cancer
  - the lifetime of a light bulb
  - the waiting and service time for a customer at a coffee shop
  - the time between births at a hospital
  - the number of gallons purchased at a gas pump
  - the time to construct an office building
- A continuous random variable X with pdf

$$f(x) = \lambda e^{-\lambda x}, \qquad x > 0$$

for some real constant $\lambda > 0$ is an Exponential(λ) random variable.

$X \sim \text{Exponential}(\lambda)$

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"rate"

CDF

$$F_X(x) = P(X \le x) = \int_0^x f(w)\,dw = \int_0^x \lambda e^{-\lambda w}\,dw = -e^{-\lambda w}\Big|_0^x$$

$$= 1 - e^{-\lambda x}, \quad x > 0$$

$$F_X(x) = \begin{cases} 0 & , x \le 0 \\ 1 - e^{-\lambda x} & , x > 0 \end{cases}$$

Memoryless Property

For $X \sim \mathrm{Exp}(\lambda)$ and any two positive real numbers x and y

$$P(X \geq x + y | X \geq x) = P(X \geq y).$$

Proof: $P(X \geq x+y \mid X \geq x) = \dfrac{P(X \geq x+y \text{ AND } X \geq x)}{P(X \geq x)}$

$$= \frac{P(X \geq x+y)}{P(X \geq x)} = \frac{1 - P(X \leq x+y)}{1 - P(X \leq x)}$$

$$= \frac{1 - F_X(x+y)}{1 - F_X(x)} = \frac{\cancel{1-(1-}e^{(x+y)\lambda})}{\cancel{1-(1-}e^{-\lambda x})}$$

$$= e^{-\lambda y} = 1-(1-e^{-\lambda y}) = 1 - F_X(y)$$

$$= P(X \geq y) \quad \square$$

Theorem

$\Gamma(k) = \int_0^\infty x^{(k-1)} \boxed{e^{-x}} dx$

Theorem

If $X \sim \mathrm{Exp}(\lambda)$, then

$$E\left(X^s\right) = \frac{\Gamma(s+1)}{\lambda^s}, \quad s > -1$$

where $\Gamma(k) = \int_0^\infty x^{k-1} e^{-x} dx$.

$x = \frac{1}{\lambda} y \qquad dx = \frac{1}{\lambda} dy$

$$\underbrace{E(X^s)}_{g(X)} = \int_0^\infty x^s \lambda e^{-\lambda x} dx \overset{\text{let } y = \lambda x}{=\!=} \int_0^\infty \left(\frac{1}{\lambda} y\right)^s \cancel{\lambda} e^{-y} \frac{1}{\cancel{\lambda}} dy$$

$$= \int_0^\infty \boxed{\frac{1}{\lambda^s}} y^s e^{-y} dy = \frac{\int_0^\infty y^s e^{-y} dy}{\lambda^s} = \frac{\Gamma(s+1)}{\lambda^s}$$

Notes on Gamma function

- When k is an integer, $\Gamma(k) = (k-1)!$
- The gamma function is minimized at $k \cong 1.4616$
- $\Gamma(k+1) = k\Gamma(k)$ for $k > 0$
- $\Gamma\left(\frac{1}{2}\right) = \sqrt(\pi)$
- The gamma function is calculated in R with *gamma*
- The gamma function is calculated in Maple with *GAMMA*
- $E\left(X^s\right) = \frac{s!}{\lambda^s}$

$$\Gamma(s+1) = \int_0^\infty t^s e^{-t}\,dt = \underbrace{\int_0^\infty \frac{e^{-t}}{s+1}\,dt^{s+1}}$$

$\int u\,dv = uv - \int v\,du$

$$= \frac{e^{-t}}{s+1} t^{s+1} \Big|_0^\infty - \frac{1}{s+1}\int_0^\infty t^{s+1}\,de^{-t}$$

$$= +\frac{1}{s+1}\int_0^\infty e^{s+1} e^{-t}\,dt$$

$$= \frac{1}{s+1}\,\Gamma(s+2)$$

$$\Gamma(s+2) = (s+1)\Gamma(s+1)$$

$$\Gamma(k) = \int_0^\infty x^{k-1} e^{-x}\,dx$$

$$\Gamma(1) = \int_0^\infty x^{1-1} e^{-x}\,dx$$

$$= \int_0^\infty e^{-x}\,dx$$

$$= -e^{-x}\Big|_0^\infty$$

$$= 1$$

$$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

$$\Gamma(k) = (k-1)\Gamma(k-1)$$ k is an integer

$$= (k-1)(k-2)\Gamma(k-2)$$

$$= (k-1)(k-2)(k-3)\Gamma(k-3) = \cdots$$

$$= \underbrace{(k-1)(k-2)(k-3)\cdots}\ 1 \cdot \overset{1}{\Gamma(1)}$$

$$= (k-1)!$$

Mean

$$E(X^s) = \frac{\Gamma(s+1)}{\lambda^s}$$

$$\mu = E(X) \overset{s=1}{=\!=} \frac{\Gamma(2)}{\lambda^1} = \frac{(2-1)!}{\lambda} = \frac{1}{\lambda}$$

Variance

$$V(X) = E(X^2) - \{E(X)\}^2$$

$$E(X^2) = \frac{\Gamma(2+1)}{\lambda^2} = \frac{(3-1)!}{\lambda^2} = \frac{2}{\lambda^2}$$

$$V(X) = \frac{2}{\lambda^2} - \left(\frac{1}{\lambda}\right)^2 = \frac{1}{\lambda^2}$$

$$M_X(t) = E(\underbrace{e^{tX}}_{g(x)}) = \int_0^\infty \lambda e^{tx} e^{-\lambda x}\, dx$$

$$= \int_0^\infty \lambda\, e^{-(\lambda-t)x}\, dx$$

$$= -\frac{\lambda}{\lambda-t} e^{-(\lambda-t)x}\Big|_0^\infty$$

$$= \frac{\lambda}{\lambda-t}, \quad t<\lambda$$

$$E(X) = \frac{dM_X(t)}{dt}\Big|_{t=0}$$

$$= \frac{+\lambda}{(\lambda-t)^2}\Big|_{t=0} = \frac{1}{\lambda}$$

$$E(X^2) = \frac{d^2M_X(t)}{dt}\Big|_{t=0}$$

$$= \frac{2\lambda}{(\lambda-t)^3}\Big|_{t=0}$$

$$= \frac{2}{\lambda^2}$$

Skewness

$$E\left\{\left(\frac{X-\mu}{\sigma}\right)^3\right\} = 2$$

Kurtosis

$$E\left\{\left(\frac{x-\mu}{\delta}\right)^4\right\} = 9$$

R Functions

Function	Returned Value
dunif(x, λ) exp	calculates the probability density function $f(x)$
punif(x, λ) exp	calculates the cumulative distribution function $F(x)$
qunif(u, λ) exp	calculates the percentile (quantile) $F^{-1}(u)$
runif(m, λ) exp	generates m random variates

Example 1

Let $X \sim \mathrm{Exp}(\lambda)$, what is the distribution of $Y = \lfloor X \rfloor$?

$$X > 0 \Rightarrow \begin{cases} Y = 0 & 0 < x < 1 \\ Y = 1 & 1 \le x < 2 \\ Y = 2 & 2 \le x < 3 \\ \vdots & \end{cases}$$

$Y \sim \mathrm{Geo}(1 - e^{-\lambda})$

$$P(Y=0) = P(0 < X < 1) = \int_0^1 \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_0^1 = 1 - e^{-\lambda}$$

$$P(Y=1) = P(1 \le X < 2) = \int_1^2 \lambda e^{-\lambda x}\,dx = -e^{-\lambda x}\Big|_1^2 = e^{-\lambda} - e^{-2\lambda}$$

$$P(Y=2) = P(2 \le X < 3) = \int_2^3 \lambda e^{-\lambda x}\,dx = e^{-2\lambda} - e^{-3\lambda}$$

$$\cdots$$

$$P(Y=y) = P(y \le X < y+1) = \int_y^{y+1} \lambda e^{-\lambda x}\,dx = e^{-\lambda y} - e^{-\lambda(y+1)}$$

$$= \underbrace{(1 - e^{-\lambda})\, e^{-\lambda y}} = \underbrace{(1 - e^{-\lambda})}_{p}\,(\underbrace{e^{-\lambda}}_{1-p})^{y}$$

Summary

The exponential distribution

- is the fundamental distribution with positive support
- has a single positive parameter λ
- is the only continuous distribution with the memoryless property
- has pdf

$$f(x) = \lambda e^{-\lambda x}, \quad x > 0$$

- has moments

$$\mu = \frac{1}{\lambda} \quad \text{and} \quad \sigma^2 = \frac{1}{\lambda^2}$$

- has a second parameterization: $\theta = \frac{1}{\lambda}$

$$f(x) = \frac{1}{\theta} e^{\frac{-x}{\theta}}, \quad x > 0$$

and the population mean and variance are

$$\mu = \theta \quad \text{and} \quad \sigma^2 = \theta^2$$

THANK YOU!