Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 5. Common Continuous Distribution

5.1 Uniform Distribution

$\mathcal{X}$, pdf, cdf
μ, σ^2, MGF

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Parameters

Parameters

- 1 **Location** (or shift) parameters
- 2 **Scale** parameters are used to expand or contract the x-axis by a factor
- 3 **Shape** parameters affect the shape of the probability density function

Uniform Distribution

Uniform Distribution

- A continuous random variable X is "uniformly distributed" between a and b if X is equally likely to assume any value on the interval (a, b).
- A continuous random variable X with pdf

$X \sim U(a,b)$

$$f(x) = \frac{1}{b-a}, \qquad a < x < b$$

for real constants a and b satisfying $a < b$ is a $U(a, b)$ random variable.

$\mathcal{S} = \{ a < x < b \}$

CDF

$$F_X(x) = P(X \le x) = \int_a^x f(x)\,dx = \int_a^x \frac{1}{b-a}\,dx$$

$$= \frac{x}{b-a}\Big|_a^x = \frac{x-a}{b-a} \qquad a < x < b.$$

$$F_X(x) = \begin{cases} 0 & x \le a \\ \dfrac{x-a}{b-a} & a < x < b \\ 1 & x \ge b \end{cases}$$

Mean

$$\mu = E(X) = \int_a^b x f(x)\,dx = \int_a^b x \frac{1}{b-a}\,dx$$

$$= \frac{x^2}{2(b-a)}\Bigg|_a^b = \frac{b^2 - a^2}{2(b-a)} = \frac{(b+a)\cancel{(b-a)}}{2\cancel{(b-a)}} = \frac{a+b}{2}$$

Variance

$$V(X) = E(\underset{?}{X^2}) - \{E(X)\}^2$$

$$E(\underbrace{X^2}_{g(X)}) = \int_a^b x^2 \frac{1}{b-a}\, dx = \frac{x^3}{3(b-a)}\Bigg|_a^b = \frac{b^3-a^3}{3(b-a)}$$

$$= \frac{a^2+ab+b^2}{3}$$

$$V(X) = \frac{a^2+ab+b^2}{3} - \left(\frac{a+b}{2}\right)^2 = \frac{\cancel{4a}^2+4ab+4b^2-3a^2-6ab-3b^2}{12}$$

$$= \frac{a^2-2ab+b^2}{12} = \frac{(b-a)^2}{12}$$

MGF

$$M_X(t) = E(\underbrace{e^{tX}}_{g(X)}) = \int_a^b e^{tx} \frac{1}{b-a}\,dx = \left.\frac{e^{tx}}{t(b-a)}\right|_a^b$$

$$= \frac{e^{bt} - e^{at}}{t(b-a)}, \quad t \neq 0$$

Skewness

$$E\left\{\left(\frac{X-\mu}{\sigma}\right)^3\right\} = 0$$

Kurtosis

$$E\left\{\left(\frac{X-\mu}{\delta}\right)^4\right\} = 9/5$$

R Functions

R Functions

Function	Returned Value
dunif(x, a, b)	calculates the probability ~~mass~~ density function $f(x)$
punif(x, a, b)	calculates the cumulative distribution function $F(x)$
qunif(u, a, b)	calculates the percentile (quantile) $F^{-1}(u)$
runif(m, a, b)	generates m random variates

Example 1

A shuttle train at a busy airport completes a circuit between two terminals every five minutes. What is the probability that a passenger will wait more than three minutes for a shuttle train?

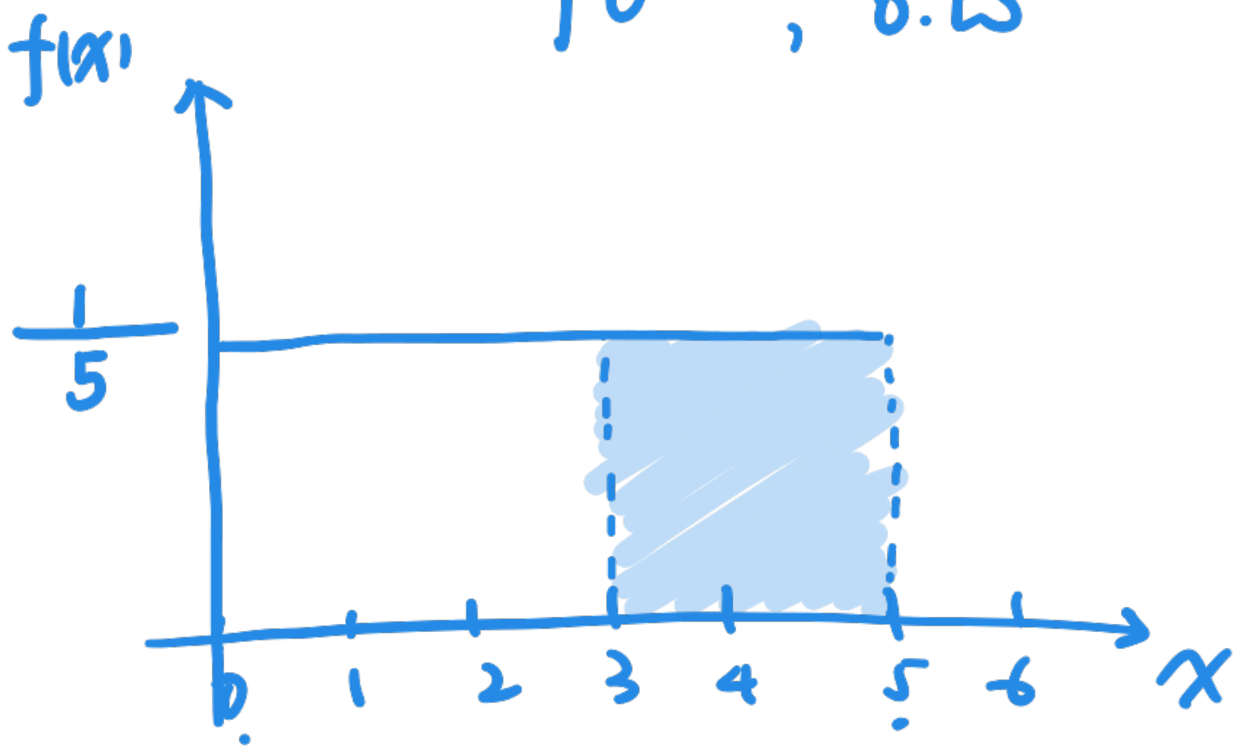
Let X = waiting time for a passenger.

$X \sim U(0,5)$

$$f(x) = \begin{cases} \frac{1}{5}, & 0<x<5 \\ 0, & \text{o.w} \end{cases}$$

$$P(X>3) = \int_3^5 f(x)\,dx$$

$$= \int_3^5 \frac{1}{5}\,dx = \frac{2}{5}$$



Example 2

What is the probability that the quadratic equation $x^2 + Bx + 1 = 0$ has two real roots, where $B \sim U(0, 3)$? (Hint: The quadratic equation $ax^2 + bx + c = 0$ has two real roots if the discriminant $b^2 - 4ac$ is positive.)

$B^2 - 4 \cdot 1 \cdot 1 > 0 \Rightarrow B^2 > 4 \Rightarrow B > 2$ OR $B < -2$.

$B \sim U(0,3) \Rightarrow f_B(b) = \begin{cases} \frac{1}{3}, & 0 < b < 3 \\ 0, & \text{o.w} \end{cases}$

$P(B > 2) = \int_2^3 \frac{1}{3}\, db = \frac{1}{3}$

Example 3

Let $X \sim U(0,1)$, find $V(3\lfloor 2X \rfloor + 4)$.

floor

$\lfloor 1.4 \rfloor = 1$

$\lfloor 2.85 \rfloor = 2$

$X \sim U(0,1) \Rightarrow f(x) = \begin{cases} 1. & 0<x<1 \\ 0. & \text{o.w} \end{cases}$

$V(Z) = E(Z^2) - \{E(Z)\}^2$

$E(Z) = 0 \times \frac{1}{2} + 1 \times \frac{1}{2} = \frac{1}{2}$

$E(Z^2) = 0^2 \times \frac{1}{2} + 1^2 \times \frac{1}{2} = \frac{1}{2}$

$Y = 2X \Rightarrow 0 < y < 2$

$Z = \lfloor Y \rfloor = \lfloor 2X \rfloor \Rightarrow \begin{cases} 0, & 0<y<1 \quad (0<x<0.5) \\ 1, & 1 \le y < 2 \quad (0.5 \le x < 1) \end{cases}$

$P(Z=0) = P(0<Y<1) = P(0<X<0.5) = \int_0^{0.5} 1\, dx = \frac{1}{2}$

$P(Z=1) = P(1 \le Y < 2) = P(0.5 \le X < 1) = \int_{0.5}^{1} 1\, dx = \frac{1}{2}$

$V(3Z+4) = V(3Z) = 3^2 V(Z) = 3^2 \left\{ \frac{1}{2} - \left(\frac{1}{2}\right)^2 \right\} = 9 \left\{ \frac{1}{2} - \frac{1}{4} \right\} = \frac{9}{4}$

$$X \sim U(0,1) \Rightarrow M_X(t) = \frac{e^{1t} - e^{0t}}{t(1-0)} = \frac{e^{t}-1}{t} \qquad t \neq 0$$

$$Y = 2X \Rightarrow M_Y(t) = E(e^{tY}) = E(e^{2Xt}) = M_X(2t)$$

$$= \frac{e^{2t} - 1}{2t} = \frac{e^{2t} - e^{0t}}{t(2-0)} \qquad t \neq 0$$

$$\Rightarrow Y \sim U(0,2)$$

Example 4

Divide a line segment of unit length randomly into two parts. Find the expected value of the product of the lengths of the two segments.

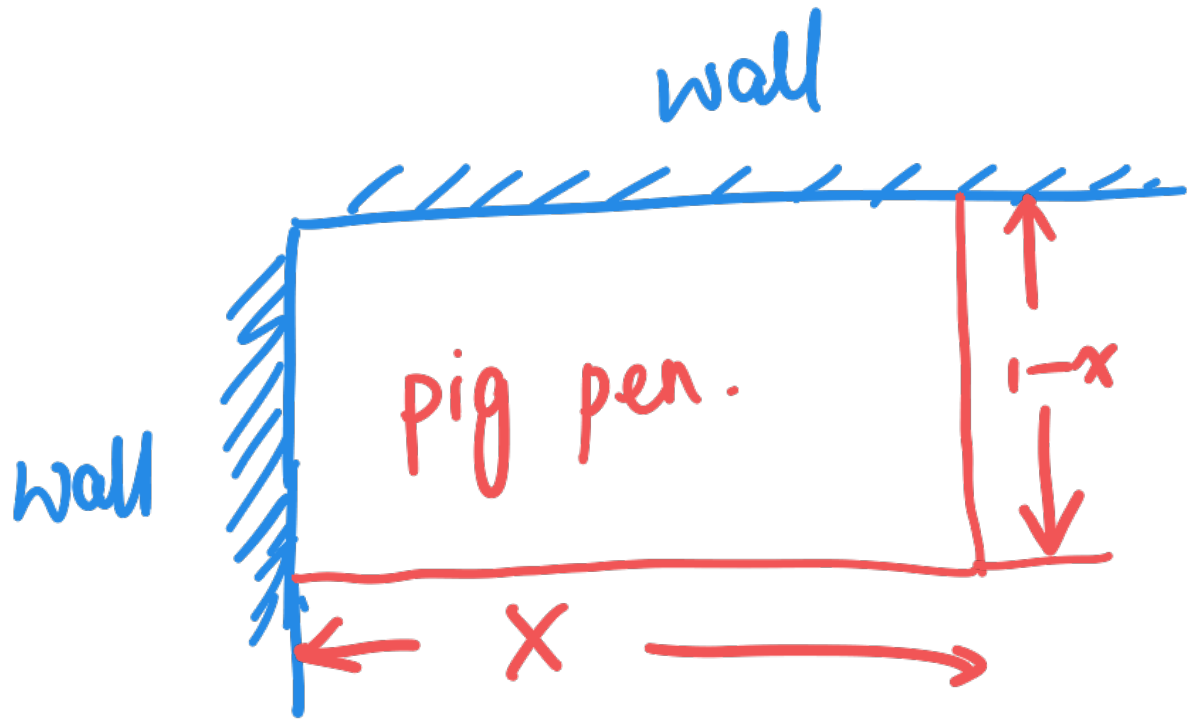
$X \sim U(0,1)$

$$\Rightarrow f_X(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$$

$$E\{\underbrace{X(1-X)}_{g(x)}\} = \int_0^1 x(1-x)\, 1\, dx$$

$$= \left(\frac{x^2}{2} - \frac{x^3}{3}\right)\Big|_0^1$$

$$= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$$



THANK YOU!