Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 4. Common Discrete Distributions

4.5 Poisson Distribution

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Approximation to Binomial Distribution

$X \sim Bin(n,p) \Rightarrow \mu = E(X) = np \Rightarrow p = \frac{\mu}{n}$

$$f_X(x) = \binom{n}{x} p^x (1-p)^{n-x} = \frac{\cancel{n!}}{x!\cancel{(n-x)}!}\left(\frac{\mu}{n}\right)^x \left(1-\frac{\mu}{n}\right)^{n-x}$$

$$= \frac{n}{n}\frac{(n-1)}{n}\frac{(n-2)}{n}\cdots\frac{(n-x+1)}{n}\frac{\mu^x}{x!}\left(1-\frac{\mu}{n}\right)^n\left(1-\frac{\mu}{n}\right)^{-x}$$

$$n\to\infty \quad \approx \frac{\mu^x}{x!}\left(1-\frac{\mu}{n}\right)^n \quad [\text{if } x \text{ is small relative to } n]$$

$$= \frac{\mu^x}{x!} e^{-\mu}$$

$\frac{\mu}{n} \to p$

$$\log\left(1-\frac{\mu}{n}\right)^n = n\log\left(1-\frac{\mu}{n}\right)$$

$$= n\left(-\frac{\mu}{n} - \frac{\mu^2}{2n^2} - \frac{\mu^3}{3n^3}\cdots\right) \approx -\mu$$

$$\left[\log(1-x) = -\sum_{n=1}^{\infty}\frac{x^n}{n}\right]$$

Poisson Distribution

Poisson Distribution

A discrete random variable X with probability mass function

$$f(x) = \frac{\lambda^x e^{-\lambda}}{x!}, \qquad x = 0, 1, 2, \ldots$$

for $\lambda > 0$ is *Poisson*(λ) random variable.

Mean

$X \sim Pois(\lambda)$ $\quad \mathcal{X} = \{0, 1, 2, \cdots\}$

$$f(x) = \underbrace{\frac{\lambda^x e^{-\lambda}}{x!}}$$

$$E(X) = \underbrace{\sum_{x=0}^{\infty}} x f(x) = \sum_{x=1}^{\infty} x f(x) = \sum_{x=1}^{\infty} \cancel{x} \boxed{\frac{\lambda^x e^{-\lambda}}{\cancel{x!}\,(x-1)!}}$$

let $\underline{y = x-1}$ $\quad \displaystyle \sum_{y=0}^{\infty} \frac{\lambda^y \boxed{\lambda} e^{-\lambda}}{y!} = \lambda \underbrace{\sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!}}_{1} = \lambda$

Variance

$$V(X) = E(X^2) - \{E(X)\}^2 = E\{\overset{?}{X(X-1)}\} + \overset{\lambda}{E(X)} - \overset{\lambda^2}{\{E(X)\}^2}$$

$$E\{\underbrace{X(X-1)}_{g(X)}\} = \sum_{x=0}^{\infty} \cancel{x(x-1)}\, \frac{\lambda^x e^{-\lambda}}{\cancel{x!}\,(x-2)!} = \sum_{x=2}^{\infty} \frac{\lambda^{x-2}\,\lambda^2\, e^{-\lambda}}{(x-2)!}$$

Let $y = x-2$

$$\sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!}\,\lambda^2 = \lambda^2 \underbrace{\sum_{y=0}^{\infty} \frac{\lambda^y e^{-\lambda}}{y!}}_{1} = \lambda^2$$

$$V(X) = \lambda^2 + \lambda - \lambda^2 = \lambda$$

MGF

$$M(t) = E(e^{tX}) = \sum_{x=0}^{\infty} e^{tx} \frac{\lambda^x e^{-\lambda}}{x!}$$

$$= \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x e^{-\lambda}}{x!}$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

$$= e^{-\lambda} \sum_{x=0}^{\infty} \frac{(e^t\lambda)^x}{x!}$$

$$= e^{-\lambda} e^{\lambda e^t} = e^{\lambda(e^t - 1)}, \quad -\infty < t < \infty$$

$$M'(t) = e^{\lambda(e^t-1)} \lambda e^t$$

$$M'(t)\Big|_{t=0} = e^{\lambda(e^0-1)} \lambda e^0 = \lambda$$

$$M''(t) = e^{\lambda(e^t-1)} \lambda^2 e^{2t} + e^{\lambda(e^t-1)} \lambda e^t$$

$$M''(t)\Big|_{t=0} = e^{\lambda(e^0-1)} \lambda^2 e^{2\cdot 0} + e^{\lambda(e^0-1)} \lambda e^0$$

$$= \lambda^2 + \lambda$$

$$V(X) = \lambda^2 + \lambda - \lambda^2 = \lambda.$$

Skewness

$$E\left\{\left(\frac{X-\mu}{\delta}\right)^3\right\} = \frac{1}{\sqrt{\lambda}}$$

Kurtosis

$$E\left\{\left(\frac{X-\mu}{\delta}\right)^4\right\} = \frac{1+3\lambda}{\lambda}$$

R Functions

Function	Returned Value
dpois(x, λ)	calculates the probability mass function $f(x)$
ppois(x, λ)	calculates the cumulative mass function $F(x)$
qpois(u, λ)	calculates the percentile (quantile) $F^{-1}(u)$
rpois(m, λ)	generates m random variates

d/p/q/rpois(x/u/m, λ)

THANK YOU!