Department of Mathematics
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MATH 451/551

Chapter 3. Random Variables

3.5 Inequalities

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Markov's Inequality

Let X be a random variable with nonnegative support (that is, $P(X \geq 0) = 1$) for which $E(X)$ exists. Then for any positive constant a

$$P(X \geq a) \leq \frac{E(X)}{a}.$$

Assume X is continuous

$$E(X) = \int_{S} x f(x)\,dx = \int_0^\infty x f(x)\,dx = \int_0^a x f(x)\,dx + \int_a^\infty x f(x)\,dx$$

$$\geq \int_a^\infty x f(x)\,dx \geq \int_a^\infty a f(x)\,dx = a\int_a^\infty f(x)\,dx = aP(X \geq a)$$

$$\therefore P(X \geq a) \leq \frac{E(X)}{a}$$

Example 1

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$\{1,2,3,4,5,6\}$

Roll a fair die 60 times. Let the random variable X be the number of sixes that appear. Use Markov's inequality to find an upper bound on the probability of rolling 30 or more sixes.

$E(X)\,? \Leftarrow f(x) = ?$ $\quad X = \#6 \Rightarrow \mathcal{A} = \{0,1,2,\cdots,60\}$

$f(x) = P(X=x) = \binom{60}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{60-x} \Rightarrow X$ is discrete.

$E(X) = \sum_{\mathcal{A}} x f(x) = \sum_{x=0}^{60} x f(x) = \sum_{x=1}^{60} x f(x) = \sum_{x=1}^{60} x \binom{60}{x}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{60-x}$

$= \sum_{x=1}^{60} \cancel{x} \frac{60!}{\cancel{x!}\,(x-1)!\,(60-x)!}\left(\frac{1}{6}\right)^x\left(\frac{5}{6}\right)^{60-x}$ $\quad$ Let $y = x-1$

$= \sum_{y=0}^{59} \frac{60!}{y!\,(60-(y+1))!}\left(\frac{1}{6}\right)^y\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^{60-(y+1)} = \sum_{y=0}^{59} \frac{59!\cdot 60\left(\frac{1}{6}\right)^y\frac{1}{6}\left(\frac{5}{6}\right)^{(59+1)-(y+1)}}{y!\,((59+1)-(y+1))!}$

$$E(X) = \sum_{y=0}^{59} \frac{60 \cdot 59! \cdot \frac{1}{6}}{y!\,(59-y)!} \left(\frac{1}{6}\right)^{y} \left(\frac{5}{6}\right)^{59-y}$$

$$= 60 \cdot \frac{1}{6} \sum_{y=0}^{59} \frac{59!}{y!\,(59-y)!} \left(\frac{1}{6}\right)^{y} \left(\frac{5}{6}\right)^{59-y}.$$

$$= 60 \cdot \frac{1}{6} = 10$$

$$P(X \geq 30) \leq \frac{E(X)}{30} = \frac{10}{30} = \frac{1}{3}.$$

Markov's Inequality Extension

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Let $g(X)$ be a nonnegative function of the random variable X. If $E\{g(X)\}$ exists, then, for every positive, real constant a,

$$P\{g(X) \geq a\} \leq \frac{E\{g(X)\}}{a}.$$

Chebyshev's Inequality

Let the random variable X have a finite population mean μ and a finite population variance σ^2. For every real-valued $k > 0$

$$P\{|X - \mu| \geq k\sigma\} \leq \frac{1}{k^2}.$$

Let $Y = (X-\mu)^2$ & $a = k^2\delta^2$

By Markov Inequality: $P(Y \geq a) \leq \frac{E(Y)}{a}$

$$P\left((x-\mu)^2 \geq k^2\delta^2\right) \leq \frac{E\{(X-\mu)^2\}}{k^2\delta^2} = \frac{V(X)}{k^2\delta^2} = \frac{\delta^2}{k^2\delta^2} = \frac{1}{k^2}$$

$$= P\left(|x-\mu| \geq k\delta\right)$$

$P(|X-\mu| \geq k\delta) \leq \frac{1}{k^2}$

Example 2

Let X be the number of screws delivered to a box by an automatic filling device. Assume $\mu = 1000$ and $\sigma^2 = 25$. There are problems associated with having too many (giving away free product) or too few (potential irritated customers) screws in a box. Use Chebyshev's inequality to find a bound on $P(994 < X < 1006)$.

$\frac{1006-1000}{\sqrt{25}} = 1.2$

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$P(994 < X < 1006)$

$= P(\mu - 1.2\delta < X < \mu + 1.2\delta)$

$= P(|X-\mu| < 1.2\delta) \geq 1 - \frac{25}{36} = \frac{11}{36}$

$P(|x-\mu| \geq 1.2\delta) \leq \frac{1}{1.2^2} = \frac{25}{36}$

THANK YOU!