Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 3. Random Variables

3.4 Moment Generating Function

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Moment Generating Functions

Moment Generate Function (MGF)

Let X be a random variable, then the **moment generating function** (MGF) of X is

$$M(t) = E\left(e^{tX}\right)$$

provided that the expected value exists on the interval $-h < t < h$ for some positive real number.

- The moment generating function at $t = 0$ must be 1, i.e. $M(0) = E\left(e^{0}\right) = 1$.

- they are good at generating moments
- they will also be used to find the distribution of sums of independent random variables
- they will also be used to find the limiting distribution of a random variable

Example 18

Example 18

Find the moment generating function for a continuous random variable X that is uniformly distributed between 0 and 1.

Example 19

Consider the random variable X with moment generating function

$$M(t) = 0.7e^{t} + 0.2e^{2t} + 0.1e^{3t}, \quad -\infty < t < \infty.$$

Is X discrete or continuous? What is the probability mass function or probability density function of X?

Moment Generating Functions

Theorem 3.8

If X has moment generating function $M(t)$ then for some positive integer r

$$E\left(X^r\right) = M^{(r)}(0) = \frac{d^r}{dt^r} M(t)\Big|_{t=0}.$$

provided that the expected value exists on the interval $-h < t < h$ for some positive real number.

Example 20

Use the moment generating function to find $E(X)$, $E\left(X^2\right)$, and $E\left(X^3\right)$ for the continuous random variable X with probability density function

$$f(x) = e^{-x}, \qquad x > 0.$$

Example 21

Use the moment generating function to find $E(X)$, $E\left(X^2\right)$, and $E\left(X^3\right)$ for the discrete random variable X with probability mass function

$$f(x) = \begin{cases} 0.7, & x = 1 \\ 0.2, & x = 2 \\ 0.1, & x = 3 \end{cases}.$$

Example 22

Consider the random variable $X \sim N(\mu, \sigma^2)$, with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \qquad -\infty < x < \infty.$$

Find the distribution for $Y = 3X + 4$.

follow

Normal

$\Rightarrow X$ is continuous

① Find $M_X(t)$

$$M_X(t) = E(e^{tX}) = \int_{\mathbb{R}} e^{tx} \frac{1}{\sqrt{2\pi\delta^2}} \exp\left\{-\frac{(x-\mu)^2}{2\delta^2}\right\} dx$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\delta^2}} \exp\left\{-\frac{x^2 - 2\mu x + \mu^2 - 2\delta^2 t x}{2\delta^2}\right\} = \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\delta^2}} \exp\left\{-\frac{x^2 - 2(\mu+\delta^2 t)x + \mu^2}{2\delta^2}\right\}$$

$$= \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\delta^2}} \exp\left[-\frac{\{x-(\mu+\delta^2 t)\}^2}{2\delta^2}\right] \exp\left\{\frac{+2\delta^2\mu t + \delta^4 t^2}{2\delta^2}\right\} dx$$

$$= \exp\left\{\frac{2\delta^2 \mu t + \delta^4 t^2}{2\delta^2}\right\} \int_{\mathbb{R}} \frac{1}{\sqrt{2\pi\delta^2}} \exp\left[-\frac{\{x-(\mu+\delta^2 t)\}^2}{2\delta^2}\right] dx$$

$$= e^{\mu t + \frac{1}{2}\delta^2 t^2}$$

② Find $M_Y(t)$ for $Y = 3X+4$

$$M_Y(t) = E(e^{tY}) = E(e^{t(3X+4)}) = E(e^{3tX} \cdot e^{4t})$$

$$= e^{4t} E(e^{3tX}) = e^{4t} M_X(3t)$$

$$= e^{4t} e^{\mu \cdot 3t + \frac{1}{2}\delta^2 (3t)^2}$$

$$= e^{(3\mu+4)t + \frac{9}{2}\delta^2 t^2} = e^{(3\mu+4)t + \frac{1}{2}(3\delta)^2 t^2}$$

$$\Rightarrow Y \sim N\left((3\mu+4), (3\delta)^2\right)$$

Let X be a R.V. with MGF $M_X(t)$.

Let $Y = aX + b$ then $M_Y(t) = e^{bt} M_X(at)$. a.b are real constants.

$$M_Y(t) = E(e^{tY}) = E\left(e^{t(ax+b)}\right) = E\left(e^{atx} \cdot e^{bt}\right)$$

$$= e^{bt} E\left(e^{atx}\right) = e^{bt} M_X(at)$$

Example 23

Let X be a random variable with pdf

$$f(x) = \begin{cases} e^{1-x} & x > 1 \\ 0 & \text{o.w.} \end{cases}.$$

X is continuous.

- Find the moment generating function (MGF) of X.
- Find $E(X)$ and $V(X)$.
- Suppose that $X_1, X_2, \ldots, X_{2n-1}, X_{2n}$ are independent random variables with PDF $f(x)$, as given above. Let $T_n = \sum_{i=1}^{2n}(-1)^{i-1}X_i$, find the moment generating function of T_n.

① $M_X(t) = E(e^{tx}) = \int_1^\infty e^{tx} e^{1-x}\,dx = \int_1^\infty e\, e^{(t-1)x}\,dx$

$= e\int_1^\infty e^{(t-1)x}\,dx = \frac{e\, e^{(t-1)x}}{t-1}\Big|_1^\infty = \frac{e^t}{1-t}, \quad t<1$

③. $$\frac{dM_x(t)}{dt} = \frac{d\frac{e^t}{1-t}}{dt} = \frac{e^t(1-t)-(-1)e^t}{(1-t)^2} = \frac{e^t(2-t)}{(1-t)^2}$$

$$E(X) = \frac{dM_x(t)}{dt}\Big|_{t=0} = 2$$

$$V(X) = E(X^2) - \mu^2$$

$$\frac{d^2M_x(t)}{dt^2} = \frac{e^t(2-t)(1-t)^2 - e^t(1-t)^2 + 2(1-t)e^t(2-t)}{(1-t)^4}$$

$$= \frac{5e^t - 4te^t + t^2e^t}{(1-t)^4}$$

$$E(X^2) = \frac{d^2M_x(t)}{dt^2}\Big|_{t=0} = 5$$

$$V(X) = E(X^2) - \mu^2 = 5 - 4 = 1$$

③. $T_n = \sum_{i=1}^{2n} (-1)^{i-1} x_i = X_1 - X_2 + X_3 - X_4 + \cdots + X_{2n-1} - X_{2n}$.

$$M_{T_n}(t) = E(e^{tT_n}) = E\left(e^{t(x_1 - x_2 + \cdots + X_{2n-1} - X_{2n})}\right)$$

$$= E\left(e^{tX_1} \cdot e^{-tX_2} \cdots\cdots e^{tX_{2n-1}} \cdot e^{-tX_{2n}}\right)$$

$$= E(e^{tX_1})\, E(e^{-tX_2}) \cdots E(e^{tX_{2n-1}})\, E(e^{-tX_{2n}})$$

$$= M_{X_1}(t)\, M_{X_2}(-t) \cdots M_{X_{2n-1}}(t)\, M_{X_{2n}}(-t)$$

$$= \frac{e^t}{1-t} \; \frac{e^{-t}}{1-(-t)} \cdots \frac{e^t}{1-t} \; \frac{e^{-t}}{1-(-t)}$$

$$= \frac{e^{nt}}{(1-t)^n} \cdot \frac{e^{-nt}}{(1+t)^n} = \frac{1}{(1-t^2)^n}$$

THANK YOU!