Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 3. Random Variables

3.4 Moment Generating Function

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Moment Generating Functions

$M_X(t)$

Moment Generate Function (MGF)

Let X be a random variable, then the **moment generating function** (MGF) of X is

$M(0) = E(e^{0 \cdot X}) = E(1) = 1$

$$M(t) = E\left(e^{tX}\right)$$

provided that the expected value exists on the interval $-h < t < h$ for some positive real number.

- The moment generating function at $t = 0$ must be 1, i.e. $M(0) = E\left(e^0\right) = 1$.

- they are good at generating moments
- they will also be used to find the distribution of sums of independent random variables
- they will also be used to find the limiting distribution of a random variable

Example 18

Find the moment generating function for a continuous random variable X that is uniformly distributed between 0 and 1.

$f(x) = 1 \qquad 0<x<1$

$M_X(t) = E(\underbrace{e^{tx}}_{g(x)}) = \int_0^1 e^{tx}\, 1\, dx = \left.\frac{e^{tx}}{t}\right|_0^1 = \frac{e^t - 1}{t}, \qquad t \neq 0$

Example 19

Consider the random variable X with moment generating function

$$M(t) = 0.7e^{t} + 0.2e^{2t} + 0.1e^{3t}, \quad -\infty < t < \infty.$$

Is X discrete or continuous? What is the probability mass function or probability density function of X?

If X is discrete.

$M_X(t) = E(e^{tX}) = \sum_{\mathcal{X}} e^{tx} f(x)$ $\Rightarrow$ X is discrete

$M_X(t) = 0.7e^{t} + 0.2e^{2t} + 0.1e^{3t} = 0.7e^{1t} + 0.2e^{2t} + 0.1e^{3t}$

$\mathcal{X} = \{1, 2, 3\}$

$$f_X(x) = \begin{cases} 0.7 & x = 1 \\ 0.2 & x = 2 \\ 0.1 & x = 3 \end{cases}$$

Moment Generating Functions

Theorem 3.8

If X has moment generating function $M(t)$ then for some positive integer r

$$E(X^r) = M^{(r)}(0) = \left.\frac{d^r}{dt^r} M(t)\right|_{t=0}.$$

provided that the expected value exists on the interval $-h < t < h$ for some positive real number.

X is continuous.

$r=1$. $\frac{dM(t)}{dt} = \frac{d}{dt}\int_{\mathbb{R}} e^{tx} f(x)\,dx = \int_{\mathbb{R}} \frac{d}{dt} e^{tx} f(x)\,dx = \int_{\mathbb{R}} xe^{tx} f(x)\,dx$

$= E(Xe^{tX})$

$\left.\frac{dM(t)}{dt}\right|_{t=0} = E(xe^{0x}) = E(X)$

$r > 1.$ $$e^{tX} = 1 + tX + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \cdots$$

$$M_X(t) = E(e^{tX}) = E\left\{1 + tX + \frac{(tX)^2}{2!} + \frac{(tX)^3}{3!} + \cdots\right\}$$

$$= 1 + tE(X) + \frac{t^2}{2!}E(X^2) + \frac{t^3}{3!}E(X^3) + \cdots$$

$$E(X^r) = \frac{d^r}{dt^r} M_X(t)\Big|_{t=0}.$$

Example 20

Use the moment generating function to find $E(X)$, $E(X^2)$, and $E(X^3)$ for the continuous random variable X with probability density function

$$f(x) = e^{-x}, \qquad x > 0. \Rightarrow X \text{ is continuous.}$$

$$M_X(t) = E(e^{tX}) = \int_0^\infty e^{tx} e^{-x}\, dx = \int_0^\infty e^{(t-1)x}\, dx = \frac{e^{(t-1)x}}{t-1}\Big|_0^\infty$$

$$= \frac{1}{1-t} \qquad t < 1$$

$$\frac{dM(t)}{dt} = \frac{d\left(\frac{1}{1-t}\right)}{dt} = \frac{1}{(1-t)^2} \qquad E(X) = \frac{d}{dt} M(t)\Big|_{t=0} = \frac{1}{(1-0)^2} = 1$$

$$\frac{d^2 M(t)}{dt^2} = \frac{d\,\frac{1}{(1-t)^2}}{dt} = \frac{2}{(1-t)^3} \qquad E(X^2) = \frac{d^2}{dt^2} M(t)\Big|_{t=0} = 2$$

$$\frac{d^3 M(t)}{dt^3} = \frac{d\,\frac{2}{(1-t)^3}}{dt} = \frac{6}{(1-t)^4} \qquad E(X^3) = \frac{d^3}{dt^3} M(t)\Big|_{t=0} = 6$$

Example 21

Use the moment generating function to find $E(X)$, $E\left(X^2\right)$, and $E\left(X^3\right)$ for the discrete random variable X with probability mass function

$$f(x) = \begin{cases} 0.7, & x = 1 \\ 0.2, & x = 2 \\ 0.1, & x = 3 \end{cases}.$$

$$M_X(t) = E(e^{tX}) = \sum_{x} e^{tx} f(x) = 0.7e^{t} + 0.2e^{2t} + 0.1e^{3t}$$

$$\frac{dM_X(t)}{dt} = 0.7e^{t} + 0.4e^{2t} + 0.3e^{3t}$$

$$E(X) = 0.7 + 0.4 + 0.3 = 1.4$$

$$\frac{d^2M_X(t)}{dt^2} = 0.7e^{t} + 0.8e^{2t} + 0.9e^{3t}$$

$$E(X^2) = 0.7 + 0.8 + 0.9 = 2.4$$

$$\frac{d^3M_X(t)}{dt^3} = 0.7e^{t} + 1.6e^{2t} + 2.7e^{3t}$$

$$E(X^3) = 0.7 + 1.6 + 2.7 = 5$$

Example 22

Consider the random variable $X \sim N(\mu, \sigma^2)$, with probability density function

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}, \qquad -\infty < x < \infty.$$

Find the distribution for $Y = 3X + 4$.

Example 23

Let X be a random variable with pdf

$$f(x) = \begin{cases} e^{1-x} & x > 1 \\ 0 & \text{o.w.} \end{cases}.$$

- Find the moment generating function (MGF) of X.
- Find $E(X)$ and $V(X)$.
- Suppose that $X_1, X_2, \ldots, X_{2n-1}, X_{2n}$ are independent random variables with PDF $f(x)$, as given above. Let $T_n = \sum_{i=1}^{2n} (-1)^{i-1} X_i$, find the moment generating function of T_n.

THANK YOU!