Department of Mathematics
College of William & Mary

# MATH 451/551

## Chapter 3. Random Variables

### 3.4 Moments

population mean $\mu$. 1st moment of origin. $\Rightarrow$ central tendency.
population variance $\sigma^2$ 2nd moment about the mean. $\Rightarrow$ dispersion.

GuanNan Wang
gwang01@wm.edu

# Population Moment

## Population Moment

For a random variable $X$ and positive integer $r$, the $r$th population moment $X$ about the origin is $E(X^r)$ when the expectation exists.

## Central Population Moment

For a random variable $X$ and positive integer $r$, the $r$th population moment $X$ about the population mean is $E\{(X-\mu)^r\}$ when the expectation exists.

$\mu = E(X) = E(X^1) \Rightarrow$ 1st moment about origin.

$\sigma^2 = E((X-\mu)^2) \Rightarrow$ 2nd moment about mean.

## Standardized Random Variable

For a random variable $X$ with population mean $\mu$ and positive population standard deviation $\sigma$, the random variable $\frac{X-\mu}{\sigma}$ is called a **standard**ized **random variable**, which has population mean 0 and population standard deviation 1 when the expectations exist.

Let $Z = g(X) = \frac{X-\mu}{\sigma}$

$$E(Z) = E\{g(X)\} = E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma}E(X-\mu) = \frac{1}{\sigma}\{E(X)-E(\mu)\}$$

$$= \frac{1}{\sigma}(\mu-\mu) = 0$$

$$V(Z) = E\{(Z-\mu_Z)^2\} = E(Z^2) = E\left\{\frac{(X-\mu)^2}{\sigma^2}\right\} = \frac{1}{\sigma^2}E\{X^2-2\mu X+\mu^2\}$$

$$V(X) = E(X^2)-\mu^2$$

$$E(X^2) = V(X)+\mu^2 = \sigma^2+\mu^2$$

$$= \frac{1}{\sigma^2}\{E(X^2) - E(2\mu X) + E(\mu^2)\}$$

$$= \frac{1}{\sigma^2}\{\cancel{\mu^2}+\sigma^2\cancel{-2\mu^2}\cancel{+\mu^2}\} = 1$$

## Population Skewness

For a random variable $X$ with population mean $\mu$ and positive population standard deviation $\sigma$, the population skewness $E\left\{\left(\frac{X-\mu}{\sigma}\right)^3\right\}$ when the expectation exists.

- The population skewness is a measure of the symmetry of a probability distribution.
- If $f(x)$ is symmetric, the population skewness is 0.
- If $f(x)$ is skewed to the right, the population skewness is positive.
- If $f(x)$ is skewed to the left, the population skewness is negative.

# Population Kurtosis



## Population Kurtosis

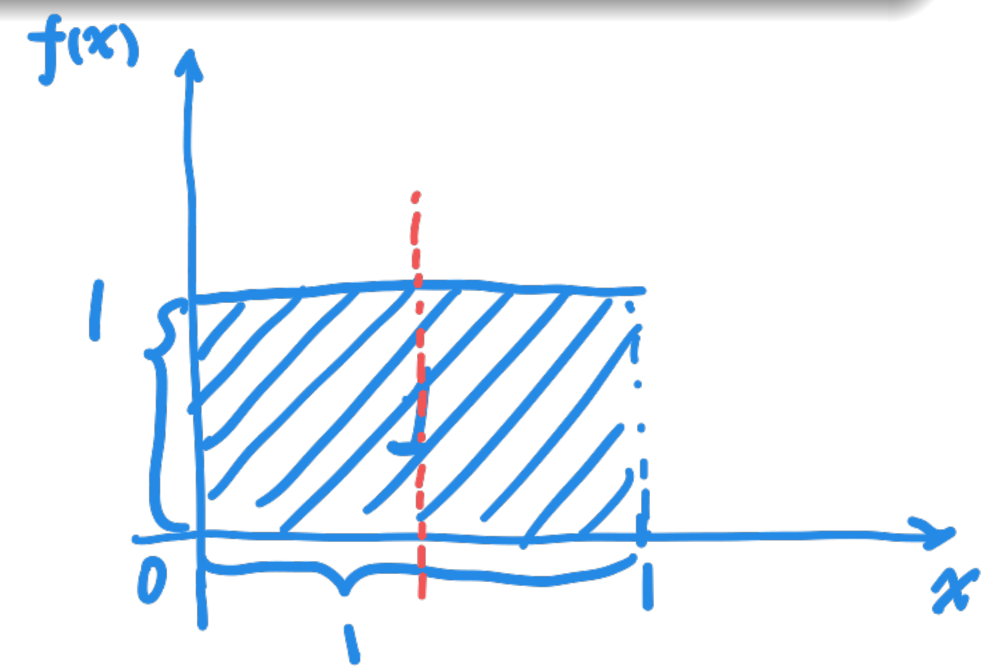
For a random variable $X$ with population mean $\mu$ and positive population standard deviation $\sigma$, the population kurtosis $E\left\{\left(\frac{X-\mu}{\sigma}\right)^4\right\}$ when the expectation exists.

- The population kurtosis is a measure of the heaviness of the tails of a probability distribution.

## Example 16

Find the population skewness and kurtosis of a continuous random variable $X$ that is uniformly distributed between 0 and 1.

$f(x) = \begin{cases} 1 & 0 < x < 1 \\ 0 & \text{o.w.} \end{cases}$ $X$ is continuous.




$\mu = E(X) = \int_0^1 x f(x)\,dx = \int_0^1 x \cdot 1\,dx = \frac{1}{2}$

$\sigma^2 = E\{(X-\mu)^2\} = E(X^2) - \mu^2$

$= \int_0^1 x^2 f(x)\,dx - \frac{1}{4} = \int_0^1 x^2\,dx - \frac{1}{4} = \frac{1}{12}$

$E\left\{\left(\frac{X-\mu}{\sigma}\right)^3\right\} = \int_0^1 \left(\frac{x-\frac{1}{2}}{\sqrt{\frac{1}{12}}}\right)^3 1\,dx = 12^{3/2} \int_0^1 (x-\frac{1}{2})^3\,dx = 0$

$E\left\{\left(\frac{X-\mu}{\sigma}\right)^4\right\} = \int_0^1 \left(\frac{x-\frac{1}{2}}{\sqrt{\frac{1}{12}}}\right)^4 1\,dx = 12^2 \int_0^1 (x-\frac{1}{2})^4\,dx = \frac{9}{5}$

## Example 17

Find the population skewness and kurtosis of a continuous random variable $X$ with probability density function

$$\underbrace{f(x) = \frac{x}{2},}\quad 0 < x < 2. \qquad X \text{ is continuous}$$

$$\mu = E(X) = \int_0^2 \frac{x^2}{2}\,dx = \frac{x^3}{6}\Big|_0^2 = \frac{4}{3}$$

$$\delta^2 = V(X) = E(X^2) - \mu^2 = \int_0^2 x^2 \frac{x}{2}\,dx - \frac{16}{9} = \frac{x^4}{8}\Big|_0^2 - \frac{16}{9} = \frac{2}{9}$$

$$E\left\{\underbrace{\left(\frac{X-\mu}{\delta}\right)^3}\right\} = \int_0^2 \left(\frac{x-\frac{4}{3}}{\sqrt{\frac{2}{9}}}\right)^3 \cdot \frac{x}{2}\,dx = -\frac{2\sqrt{2}}{15}$$

$$E\left\{\left(\frac{X-\mu}{\delta}\right)^4\right\} = \int_0^2 \left(\frac{x-\frac{4}{3}}{\sqrt{\frac{2}{9}}}\right)^4 \frac{x}{2}\,dx = \frac{12}{5}$$

① $f(x)$ & $F(x)$ completely define the distribution.

② population moments $\left\{ \begin{array}{l} \text{center.} \\ \text{spread out} \\ \text{symmetry.} \\ \text{heavy tail} \end{array} \right.$

THANK YOU!