Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 3. Random Variables

3.4 Variance

GuanNan Wang
gwang01@wm.edu

Moments

Variance

For a random variable X with population mean μ, the **population variance** of X is $\sigma^2 = V(X) = E\{(X-\mu)^2\}$ when the expected values exist.

- The units on the population variance are square of the units of the random variable X.
- The positive square root of the population variance is the **population standard deviation** σ. One reason for the popularity of σ is that its units are the same units as the random variable X.
- If the discrete random variable X has a support $\mathcal{A}$ that contains only one x-value, then $\sigma^2 = 0$. This distribution is often known as a **degenerate** distribution.
- Some authors use $V(X)$ for the population variance.
- The population variance is not the only measure of dispersion for a random variable X.
  - the population range $R = \sup(\mathcal{A}) - \inf(\mathcal{A})$,
  - the population mean absolute deviation $E(|X-\mu|)$.

Moments

Variance

For the random variable X with population mean μ and population variance σ^2, $V(X) = E(X^2) - \mu^2$, when is known as the **shortcut formula** for computing the population variance.

Example 11

Using the defining and computational formulas, find the population variance of the number of spots showing when rolling a fair die.

Example 12

Example 12

Calculate the population variance of the random variable X with probability density function

$$f(x) = \frac{x}{2}, \quad 0 < x < 2$$

using the defining formula and the computation formula.

Properties of Variance

Property 7

For the random variable X with population mean μ and population variance σ^2,

$$V(aX + b) = a^2 V(X)$$

for real constants a and b.

Example 13

Random variable X has the probability density function

$$f(x) = \begin{cases} \frac{3}{2}x^2 + x, & 0 \le x \le 1 \\ 0, & \text{o.w.} \end{cases}.$$

X is continuous.

Find $V(X)$.

$V(X) = E(X^2) - \mu^2$

① $\mu = E(X) = \int_0^1 x\left(\frac{3}{2}x^2 + x\right) dx = \int_0^1 \frac{3}{2}x^3 + x^2\, dx$

$= \frac{3}{8}x^4\Big|_0^1 + \frac{x^3}{3}\Big|_0^1 = \frac{17}{24}$

② $E(X^2) = \int_0^1 x^2\left(\frac{3}{2}x^2 + x\right) dx = \int_0^1 \frac{3}{2}x^4 + x^3\, dx$

$= \frac{3}{10} + \frac{1}{4} = \frac{11}{20}$

③ $V(X) = E(X^2) - \mu^2 = \frac{11}{20} - \left(\frac{17}{24}\right)^2 \approx 0.0487.$

Example 14

Consider the random variable X with cumulative distribution function

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{x}{5} + \frac{3}{10}, & -1 \le x < 0 \\ \frac{2}{5}, & 0 \le x < 1 \\ \frac{x}{5} + \frac{2}{5}, & 1 \le x < 3 \\ 1, & x \ge 3 \end{cases}.$$

Find $E(X)$ and $V(X)$.

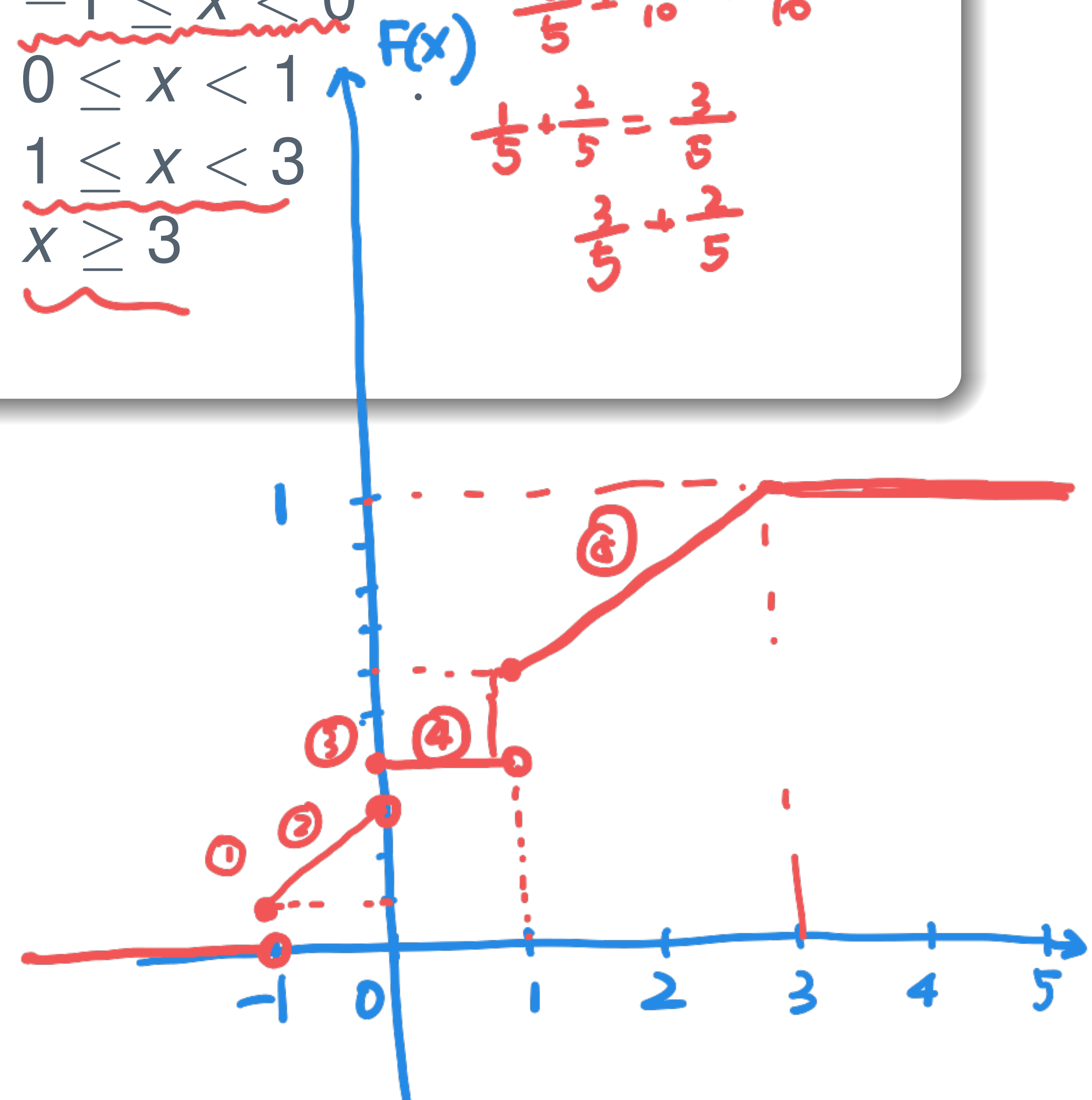
$-\frac{1}{5} + \frac{3}{10} = \frac{1}{10}$

$\frac{0}{5} + \frac{3}{10} = \frac{3}{10}$

$\frac{1}{5} + \frac{2}{5} = \frac{3}{5}$

$\frac{3}{5} + \frac{2}{5}$

$$f(x) = \begin{cases} \frac{1}{10} & x = -1 \\ \frac{1}{5} & -1 < x < 0 \\ \frac{1}{10} & x = 0 \\ \frac{1}{5} & x = 1 \\ \frac{1}{5} & 1 < x < 3 \\ 0 & \text{o.w.} \end{cases}$$



$$E(X) = \frac{1}{10} \times (-1) + \int_{-1}^{0} \frac{x}{5}\,dx + \frac{1}{10} \times (0) + \frac{1}{5} \times (1) + \int_{1}^{3} \frac{x}{5}\,dx$$

$$= -\frac{1}{10} + \frac{x^2}{10}\Big|_{-1}^{0} + 0 + \frac{1}{5} + \frac{x^2}{10}\Big|_{1}^{3} = -\frac{1}{10} - \frac{1}{10} + \frac{1}{5} + \frac{8}{10} = \frac{4}{5}$$

$$E(X^2) = \frac{1}{10} \times (-1)^2 + \int_{-1}^{0} \frac{x^2}{5}\,dx + \frac{1}{10} \times (0^2) + \frac{1}{5} \times (1^2) + \int_{1}^{3} \frac{x^2}{5}\,dx$$

$$= \frac{1}{10} + \frac{x^3}{15}\Big|_{-1}^{0} + 0 + \frac{1}{5} + \frac{x^3}{15}\Big|_{1}^{3} = \frac{1}{10} + \frac{1}{15} + \frac{1}{5} + \frac{26}{15} = \frac{21}{10}$$

$$V(X) = E(X^2) - \{E(X)\}^2 = \frac{21}{10} - \frac{16}{25} = \frac{73}{50}$$

Example 15

Let Y be a random variable of the continuous type with PDF $f(y)$, which is positive provided $0 < y < b < 1$, and is equal too zero elsewhere. Show that

$$E(Y) = \int_0^b \{1 - F(y)\}dy$$

where $F(y)$ is the cumulative distributioon function of Y.

$$E(Y) = \int_0^b y f(y)\,dy$$
$$= \int_0^b y\, d F(y)$$
$$= yF(y)\Big|_0^b - \int_0^b F(y)\,dy$$
$$= b * 1 - 0 - \int_0^b F(y)\,dy = b - \int_0^b F(y)\,dy = \int_0^b 1\,dy - \int_0^b F(y)\,dy$$

integration by parts

$$\int u\,dv = uv - \int v\,du.$$

$$f(y) = \frac{dF(y)}{dy}$$

THANK YOU!