Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 3. Random Variables

3.4 Variance

center
① expected value (mean)

$$E(X) = \begin{cases} \sum_{x} x f(x) & \text{if } X \text{ is discrete} \\ \int_{x} x f(x) dx & \text{if } X \text{ is continuous.} \end{cases}$$

② median

$$\int_{-\infty}^{\tilde{x}} f(x)\, dx = \frac{1}{2}$$

③ mode: value with highest frequence (if uniquely exist)

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Moments

spread out

$E\{(X-\mu)^2\}$

μ

Variance

For a random variable X with population mean μ, the **population variance** of X is $\sigma^2 = V(X) = E\{(X-\mu)^2\}$ when the expected values exist.

$\sigma^2 = V(X)$ ≥ 0

$\sigma \geq 0$

- The units on the population variance are square of the units of the random variable X.
- The positive square root of the population variance is the **population standard deviation** σ. One reason for the popularity of σ is that its units are the same units as the random variable X. σ
- If the discrete random variable X has a support $\mathcal{A}$ that contains only one x-value, then $\sigma^2 = 0$. This distribution is often known as a **degenerate** distribution. $\sigma = 0$
- Some authors use $V(X)$ for the population variance.
- The population variance is not the only measure of dispersion for a random variable X.
  - the population range $R = \sup(\mathcal{A}) - \inf(\mathcal{A})$,
  - the population mean absolute deviation $E(|X-\mu|)$. $E(|X-\mu|)$

Variance

For the random variable X with population mean μ and population variance σ^2, $V(X) = E(X^2) - \mu^2$, when is known as the **shortcut formula** for computing the population variance.

$V(X) = E\{(X-\mu)^2\}$

$= E\{X^2 - 2\mu X + \mu^2\} = E(X^2) - E(\overbrace{2\mu}^{c} X) + E(\mu^2)$

$= E(X^2) - 2\mu E(X) + \mu^2$

$= E(X^2) - 2\mu\mu + \mu^2$

$= E(X^2) - \mu^2$

2nd population moment.

Example 11

Using the defining and computational formulas, find the population variance of the number of spots showing when rolling a fair die.

$S = \{1, 2, 3, 4, 5, 6\}$

$f(x) = \frac{1}{6}, \quad x \in S$

$\mu = E(X) = \sum_{S} x f(x) = \frac{1+2+3+4+5+6}{6} = \frac{7}{2}$

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
$(x-\mu)^2$	$\frac{25}{4}$	$\frac{9}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{9}{4}$	$\frac{25}{4}$

$E\{(X-\mu)^2\} = \sum_{S} (x-\mu)^2 f(x)$

$= \frac{25+9+1+1+9+25}{24}$

$= \frac{35}{12}$

$E(X^2)$

x	1	2	3	4	5	6
$f(x)$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$	$\frac{1}{6}$
x^2	1	4	9	16	25	36

$E(X^2) = \frac{1+4+9+16+25+36}{6}$

$\sigma = E(X^2) - \mu^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$

Example 12

Calculate the population variance of the random variable X with probability density function

$$f(x) = \frac{x}{2}, \quad 0 < x < 2$$

using the defining formula and the computation formula.

$\mu = \int_{\Omega} x f(x)\,dx = \int_0^2 x \frac{x}{2}\,dx = \frac{x^3}{6}\Big|_0^2 = \frac{8}{6}$

$= \frac{4}{3}$

$\sigma^2 = E\{(X-\mu)^2\} = \int_{\Omega} (x - \frac{4}{3})^2 \frac{x}{2}\,dx$

$= \int_0^2 \left(\frac{x^3}{2} - \frac{4x^2}{3} + \frac{\cancel{16}^{8} x}{\cancel{18}\, 9}\right) dx$

$= \frac{x^4}{8}\Big|_0^2 - \frac{4x^3}{9}\Big|_0^2 + \frac{4x^2}{9}\Big|_0^2$

$= \frac{16}{8} - \frac{32}{9} + \frac{16}{9}$

$= \frac{16}{8} - \frac{16}{9} = \frac{2}{9}$

$E(X^2) = \int_{\Omega} x^2 \frac{x}{2}\,dx = \int_0^2 \frac{x^3}{2}\,dx$

$= \frac{x^4}{8}\Big|_0^2 = 2$

$\sigma^2 = E(X^2) - \mu^2 = 2 - \frac{16}{9} = \frac{2}{9}$

Properties of Variance

Property 7

For the random variable X with population mean μ and population variance σ^2,

$$V(aX + b) = a^2 V(X)$$

for real constants a and b.

$$V(\underbrace{aX+b}_{Y}) = E\left[\{(aX+b) - E(aX+b)\}^2\right] = E\left\{\left[aX+b - \{E(aX)+E(b)\}\right]^2\right\}$$

$$= E\left[\{aX+\cancel{b} - aE(X)\}^{2} - \cancel{b}\right]$$

$$= E\left[a^2\{X - E(X)\}^2\right] = a^2 E\left[\{X - E(X)\}^2\right]$$

$$= a^2 V(X)$$

① $b = 0$

$V(aX) = a^2 V(X)$

② $a = 0$

$V(b) = 0$

Example 13

Random variable X has the probability density function

$$f(x) = \begin{cases} \frac{3}{2}x^2 + x, & 0 \le x \le 1 \\ 0, & \text{o.w.} \end{cases}.$$

Find $V(X)$.

Example 14

Consider the random variable X with cumulative distribution function

$$F(x) = \begin{cases} 0, & x < -1 \\ \frac{x}{5} + \frac{3}{10}, & -1 \leq x < 0 \\ \frac{2}{5}, & 0 \leq x < 1 \\ \frac{x}{5} + \frac{2}{5}, & 1 \leq x < 3 \\ 1, & x \geq 3 \end{cases}.$$

Find $E(X)$ and $V(X)$.

Example 15

Let Y be a random variable of the continuous type with PDF $f(y)$, which is positive provided $0 < y < b < 1$, and is equal too zero elsewhere. Show that

$$E(Y) = \int_0^b \{1 - F(y)\} dy,$$

where $F(y)$ is the cumulative distributioon function of Y.

Thank You

THANK YOU!