Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 3. Random Variables

3.4 Properties of Expected Values

GuanNan Wang
gwang01@wm.edu

Properties of Expected Values

Property 1

Let X be a ranndom variable defined on the support $\mathcal{A}$ with probability mass function $f(x)$ if X is discrete and probability density function $f(x)$ if X is continuous. The expected value of $g(X)$ is

$$\mathrm{E}\{g(X)\} = \begin{cases} \sum_{\mathcal{A}} g(x)f(x) & X \text{ is discrete} \\ \int_{\mathcal{A}} g(x)f(x)dx & X \text{ is continuous} \end{cases}$$

when the sum of integral exists. When the sum or integral diverges, the expected value is undefined.

X is a discrete R.V.

one-to-one transformation $g(x) \Rightarrow$ let $Y = g(X)$ $= E(g(X))$

Let x_i the ith value taken by the r.v. X: $y_i = g(x_i)$

$$\sum_{\mathcal{A}} g(x)f(x) = \sum_i g(x_i)P(X=x_i) = \sum_j \sum_{i:g(x_i)=y_j} g(x_i)P(X=x_i)$$

$$= \sum_j y_j \sum_{i:g(x_i)=y_j} P(X=x_i) = \underline{\sum_j y_j P(Y=y_j)} = E(Y)$$

X

x	-1	0	1
$f(x)$	0.2	0.5	0.3

Find $E(X^2)$?

Let $Y = X^2$

x	-1	0	1
$f(x)$	0.2	0.5	0.3
$y = x^2$	1	0	1

$B = \{0, 1\}$

$P(Y=0) = P(X=0) = 0.5$

$P(Y=1) = P(X=-1) + P(X=1)$

$= 0.5$

$$E(Y) = \sum_B y\,P(Y=y)$$

$$= 0 \times 0.5 + 1 \times 0.5 = 0.5$$

Properties of Expected Values

Property 2

Given a random variable X and a real constant c

$$\mathrm{E}(c) = c.$$

Assume X is continuous. $g(X) = c$

$$E(c) = \int_{\mathcal{A}} g(x) f(x)\, dx = \int_{\mathcal{A}} c\, f(x)\, dx = c \int_{\mathcal{A}} f(x)\, dx$$

$$= c \cdot 1 = c.$$

Properties of Expected Values

Property 3

~~Let the continuous random variable X be~~ Given a random variable X and a real constant c

$$\mathrm{E}(cX) = c\mathrm{E}(X).$$

Assume X is continuous. let $g(X) = cX$

$$E(cX) = \int_{\mathcal{X}} g(x) f(x)\,dx = \int_{\mathcal{X}} cX f(x)\,dx = c\int_{\mathcal{X}} x f(x)\,dx$$

$$= c\,E(X)$$

$E(\cdot)$ linear operator.

Example 9

Let the continuous random variable X be uniformly distributed between 0 and 1 with probability density function

$$f_X(x) = 1, \quad 0 < x < 1.$$

Find $E\left(\sqrt{X}\right)$.

Let $Y = g(X) = \sqrt{X} \Rightarrow \mathcal{B} = \{0 < y < 1\}$

$F_Y(y) = P(Y \le y) = P(\sqrt{X} \le y)$

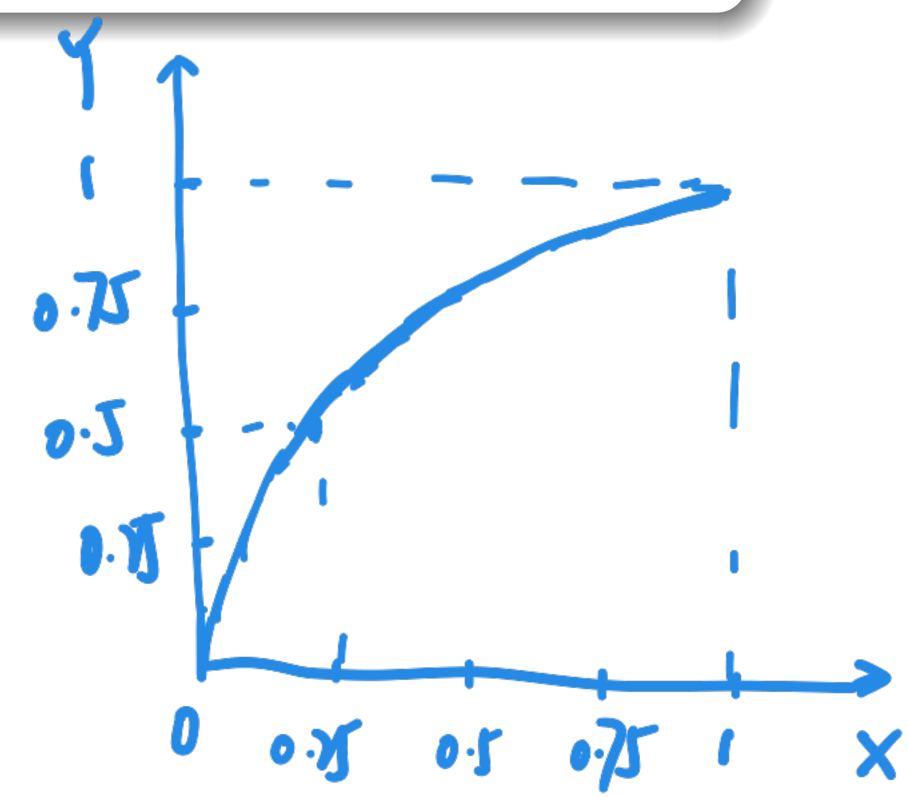
$= P(X \le y^2) = \int_{-\infty}^{y^2} f_X(x)\,dx = \int_0^{y^2} 1\,dx$

$= y^2, \quad 0 < y < 1$

$f_Y(y) = 2y, \quad 0 < y < 1$

$E(Y) = \int_{\mathcal{B}} y f_Y(y)\,dy$

$= \int_0^1 y\,2y\,dy = \frac{2}{3}$



$$E(\sqrt{X}) = \int_{\Omega} \sqrt{x}\, f(x)\, dx = \int_0^1 \sqrt{x}\, 1\, dx = \frac{2}{3} x^{3/2} \Big|_0^1 = \frac{2}{3}$$

Example 10

Example 10

Consider the discrete random variable X with probability mass function

$$f_X(x) = \frac{1}{3}, \quad x = -2, 0, 1.$$

Find $E(|X| + 4)$.

$$E(|X|+4) = \sum_{x} g(x)f(x)$$

$$= 6 \times \frac{1}{3} + 4 \times \frac{1}{3} + 5 \times \frac{1}{3}$$

$$= \frac{15}{3} = 5$$

x	-2	0	1
$f_X(x)$	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$
$g(x) = \|x\| + 4$	6	4	5

Property 4

Give a random variable X,

$$\mathrm{E}\{cg(X)\} = c\mathrm{E}\{g(X)\}.$$

when the expected values exist.

Assume X is continuous. $h(x) = c\,g(x)$

$$\underbrace{E\{cg(x)\}}_{h(x)} = \int_{\mathcal{X}} h(x)f(x)\,dx = \int_{\mathcal{X}} c\,g(x)f(x)\,dx = c\int_{\mathcal{X}} g(x)f(x)\,dx$$

$$= c\,E\{g(x)\}$$

Properties of Expected Values

Property 5

Give a random variable X and functions $g_1(X)$ and $g_2(X)$,

$$\mathrm{E}\{g_1(X) + g_2(X)\} = \mathrm{E}\{g_1(X)\} + \mathrm{E}\{g_2(X)\}.$$

when the expected values exist.

X is continuous.. $h(X) = g_1(X) + g_2(X)$

$$\mathrm{E}\{g_1(X) + g_2(X)\} = \int_{\mathcal{X}} h(x) f(x)\, dx = \int_{\mathcal{X}} \{g_1(x) + g_2(x)\} f(x)\, dx$$

$$= \int_{\mathcal{X}} g_1(x) f(x) + g_2(x) f(x)\, dx$$

$$= \int_{\mathcal{X}} g_1(x) f(x)\, dx + \int_{\mathcal{X}} g_2(x) f(x)\, dx$$

$$= \mathrm{E}\{g_1(X)\} + \mathrm{E}\{g_2(X)\}.$$

Property 5 (extension)

Give a random variable X and functions $g_1(X), g_2(X), \ldots, g_k(X)$

$$\mathrm{E}\{g_1(X) + g_2(X) + \cdots + g_k(X)\} = \mathrm{E}\{g_1(X)\} + \mathrm{E}\{g_2(X)\} + \cdots + \mathrm{E}\{g_k(X)\}.$$

THANK YOU!