Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 3. Random Variables

3.4 Expected Values

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population mean: $\mu = \frac{1}{N}\sum_{i=1}^{N} x_i$: N population. size.

7 students. X = age of their cars. 2, 4, 6, 7, 8, 4, 3

(entire group)

$\mu = \frac{2+4+6+7+8+4+3}{7} = 4.9$ parameter fixed.

sample mean: $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} x_i$: n sample size

(subset)

$\bar{x} = \frac{2+4+6+7+8+4+3}{7} = 4.9.$ statistic random

random variable: $\mathcal{A}$: all the possible values of a random experiment

(population) $X = \mathcal{A} = \{2, 3, 4, 6, 7, 8\}$

$\mu = \frac{2+4+6+7+8+4+3}{7} = 2\times\frac{1}{7} + 4\times\frac{2}{7} + 6\times\frac{1}{7} + 7\times\frac{1}{7} + 8\times\frac{1}{7} + 3\times\frac{1}{7}$

$= \sum x f(x)$ Expectation / Expected value.

Expected Values

- The mathematical expectation, or expected value, of a random variable X is what you "expect" (on the average, in the long run) for the value of X.
- The standard notatiion for the expected value of X is $\mu = \mathrm{E}(X)$.

E(X)

- The expected value of X is also known as the "population mean".

Expected Value

Let X be a random variable defined on the support $\mathcal{A}$ with probability mass function $f(x)$ if X is discrete or probability density function $f(x)$ if X is continuous.

$$\mu = E(X) = \begin{cases} \sum_{\mathcal{A}} xf(x) & X \text{ is discrete} \\ \int_{\mathcal{A}} xf(x)dx & X \text{ is continuous} \end{cases}$$

when the sum or integral exists. When the sum or integral diverges, the expected value is undefined.

Example 1

Find the expected number of spots (pips!) showing when rolling a fair die.

X = number of spots when rolling a fair die

$\mathcal{S} = \{1, 2, 3, 4, 5, 6\}$ $\Rightarrow$ discrete.

$f(x) = \frac{1}{6}$, $x = 1, 2, 3, 4, 5, 6$

$\mu = E(X) = \sum_{\mathcal{S}} x f(x) = 1\times\frac{1}{6} + 2\times\frac{1}{6} + 3\times\frac{1}{6} + 4\times\frac{1}{6} + 5\times\frac{1}{6} + 6\times\frac{1}{6} = 3.5$

average value for a long run.

Example 2

The probability mass function for the random variable X is

$$f(x) = \frac{7^x e^{-7}}{x!}, \quad x = 0, 1, 2, \ldots.$$

What is the expected value of X?

$\mathcal{X} = \{0, 1, 2, \ldots\} \Rightarrow X$ is discrete.

$$\mu = E(X) = \sum_{\mathcal{X}} x f(x) = \sum_{\mathcal{X}} x \frac{7^x e^{-7}}{x!} = \sum_{x=0}^{\infty} x \frac{7^x e^{-7}}{x(x-1)\cdots 1}$$

$$= \sum_{x=1}^{\infty} \cancel{x} \frac{7^x e^{-7}}{\cancel{x}(x-1)\cdots 1} = \sum_{x=1}^{\infty} \frac{7^x e^{-7}}{(x-1)(x-2)\cdots 1} = \sum_{x=1}^{\infty} \frac{7^x e^{-7}}{(x-1)!}$$

let $y = x-1$

$$\sum_{y=0}^{\infty} \frac{7^y \cdot 7 e^{-7}}{y!} = 7 \underbrace{\sum_{y=0}^{\infty} \frac{7^y e^{-7}}{y!}}_{=1} = 7 \times 1 = 7$$

Example 3

An urn contains 3 red balls and 4 blue balls. Balls are drawn successively at random and without replacement from the urn. Let the random variable X be the trial number when the first red ball is drawn. Find $E(X)$.

X = # trial to get the first red ball.

$S = \{1, 2, 3, 4, 5\} \Rightarrow X$ is discrete.

$f_X(x)$: $f(1) = \frac{3}{7}$ $\quad f(2) = \frac{\cancel{3}}{\cancel{6}} \times \frac{\cancel{4}^{2}}{7} = \frac{2}{7}$ $\quad f(3) = \frac{3}{5} \times \frac{\cancel{3}}{\cancel{6}} \times \frac{\cancel{4}^{2}}{7} = \frac{6}{35}$

$f(4) = \frac{3}{\cancel{4}} \times \frac{\cancel{2}}{5} \times \frac{\cancel{3}}{\cancel{6}} \times \frac{\cancel{4}}{7} = \frac{3}{35}$ $\quad f(5) = \frac{3}{3} \times \frac{1}{\cancel{4}} \times \frac{\cancel{2}}{5} \times \frac{\cancel{3}}{\cancel{6}} \times \frac{\cancel{4}}{7} = \frac{1}{35}$

$$\mu = E(X) = 1 \times \frac{3}{7} + 2 \times \frac{2}{7} + 3 \times \frac{6}{35} + 4 \times \frac{3}{35} + 5 \times \frac{1}{35} = 2$$

Example 4

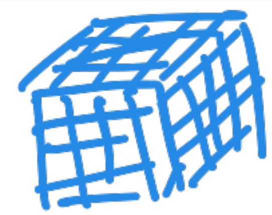
Let n be a positive integer. A cube is comprised of n^3 smaller cubes. If one of the n^3 smaller cubes is selected at random, given an expression for the expected number of exposed faces.

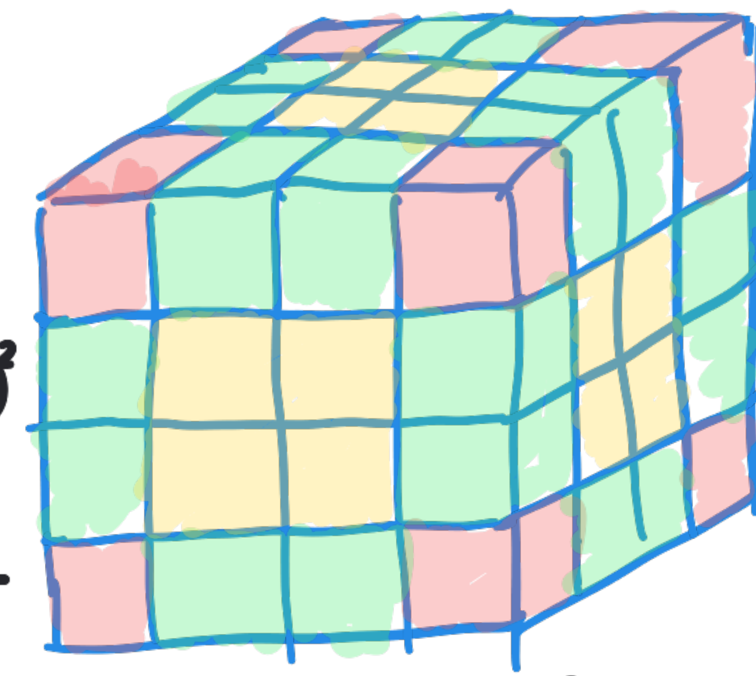
① corner: 3 faces (8)

② edge: 2 faces $(12(n-2))$

③ face: 1 face $(6(n-2)^2)$

④ interior: 0 face $(n-2)^3$





4^3 n^3

$$f(x) = \begin{cases} \frac{(n-2)^3}{n^3} & x=0 \\ \frac{6(n-2)^2}{n^3} & x=1 \\ \frac{12(n-2)}{n^3} & x=2 \\ \frac{8}{n^3} & x=3 \end{cases}$$

$$E(X) = 0 \times \frac{(n-2)^3}{n^3} + 1 \times \frac{6(n-2)^2}{n^3} + 2 \times \frac{12(n-2)}{n^3} + 3 \times \frac{8}{n^3}$$

$$= \frac{6}{n}$$

$$\lim_{n\to\infty} E(X) = \lim_{n\to\infty} \frac{6}{n} = 0.$$

Thank You

THANK YOU!