Department of Mathematics
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MATH 451/551

Chapter 3. Random Variables

3.3 Mixtures

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Mixtures

Mixtures

Let $X_1, X_2, \ldots, X_k$ be random variables drawn from distributions with cumulative distribution functions $F_{X_1}(x), F_{X_2}(x), \ldots, F_{X_k}(x)$. Let $p_1, p_2, \ldots, p_k$ be the positive mixing probabilities, where $p_1 + p_2 + \cdots + p_k = 1$. Then X has a **finite mixture distribution** if its cumulative distribution function is

$$F_X(x) = p_1 F_{X_1}(x) + p_2 F_{X_2}(x) + \cdots + p_k F_{X_k}(x).$$

- A finite mixture distribution effectively mixes k distributions according to the mixing probabilities $p_1, p_2, \ldots, p_k$ in the same fashion that a chemist mixes k solutions.
- The mixing probabilities correspond to the quantity of the solution mixed and the cumulative distribution functions model the different solutions being mixed.

Example 11

There are two factories that produce light bulbs. Factory 1 prooduces bulbs that have lifetimes modeled by the random variable X_1 with cumulative distribution function

$$F_{X_1}(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-x} & x \geq 0 \end{cases}$$

$X_1 \sim Exp(1)$

$f_{X_1}(x) = e^{-x}, \ x \geq 0$

Factory 2 produces bulbs that have lifetimes modeled by the random variable X_2 with cumulative distribution function

$$F_{X_2}(x) = \begin{cases} 0 & x < 0 \\ 1 - e^{-2x} & x \geq 0 \end{cases}$$

$X_2 \sim Exp(2)$

$f_{X_2}(x) = 2e^{-2x}, \ x \geq 0.$

Furthermore, it is known that 2/3 of the bulbs come from Factory 1 and 1/3 of the bulbs come from Factory 2. What is the probability distribution of the lifetime of a bulb whose factory of origin cannoot be identified.

$$F_{X_1}(x) = \begin{cases} 0 & x<0 \\ 1-e^{-x} & x\geq 0 \end{cases} \qquad F_{X_2}(x) = \begin{cases} 0 & x<0 \\ 1-e^{-2x} & x\geq 0. \end{cases}$$

$$P_1 = 2/3 \qquad P_2 = 1/3$$

$$F_X(x) = P_1 F_{X_1}(x) + P_2 F_{X_2}(x)$$

$$= \tfrac{2}{3}(1-e^{-x}) + \tfrac{1}{3}(1-e^{-2x}). \qquad x\geq 0$$

$$= 1 - \tfrac{2}{3}e^{-x} - \tfrac{1}{3}e^{-2x} \qquad x\geq 0.$$

$F_X(x) = P(\underbrace{X\leq x}_{A})$ { $P(A|B_1)P(B_1)$ ← light bulb from Factory 1 + $P(A|B_2)P(B_2)$ ← light bulb from Factory 2

Thank You

THANK YOU!