

**MATH 451/551**

**Chapter 3. Random Variables**  
**3.3 Transformations**

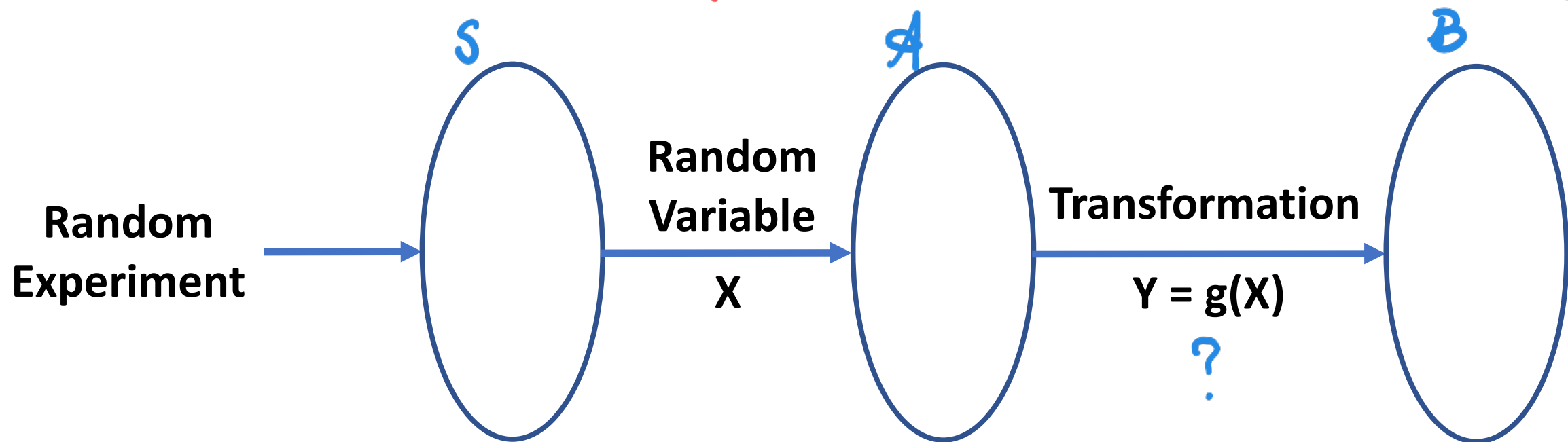
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# Transformations of Random Variables



$$F_Y(y) = P(Y \leq y) = P\{g(X) \leq y\} = P\{X \leq g^{-1}(y)\}$$



- ▶  $F_X(x)$ : cumulative distribution function for a random variable  $X$ ;
- ▶  $F_Y(y)$ : cumulative distribution function for a random variable  $Y$ ;
- ▶  $f_X(x)$ : probability mass/density function for a random variable  $X$ ;
- ▶  $f_Y(y)$ : probability mass/density function for a random variable  $Y$ .

$$F_Y(y) = P(Y \leq y) = P\{g(X) \leq y\}.$$



- ▶ We are often interested in finding the distribution of a function of a random variable.
- ▶ Our goal here is to find the cumulative distribution function of  $Y$ .

# Example 8



Let the random variable  $X$  be uniformly distributed between 0 and 1. Find the cumulative distribution function of  $Y = g(X) = \sqrt{X}$ .

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

①  $A = \{0 < x < 1\}$

$$F_X(x) = \begin{cases} 0 \\ x \\ 1 \end{cases}$$

$$\begin{aligned} x &\leq 0 \\ 0 &< x < 1 \\ x &\geq 1 \end{aligned}$$

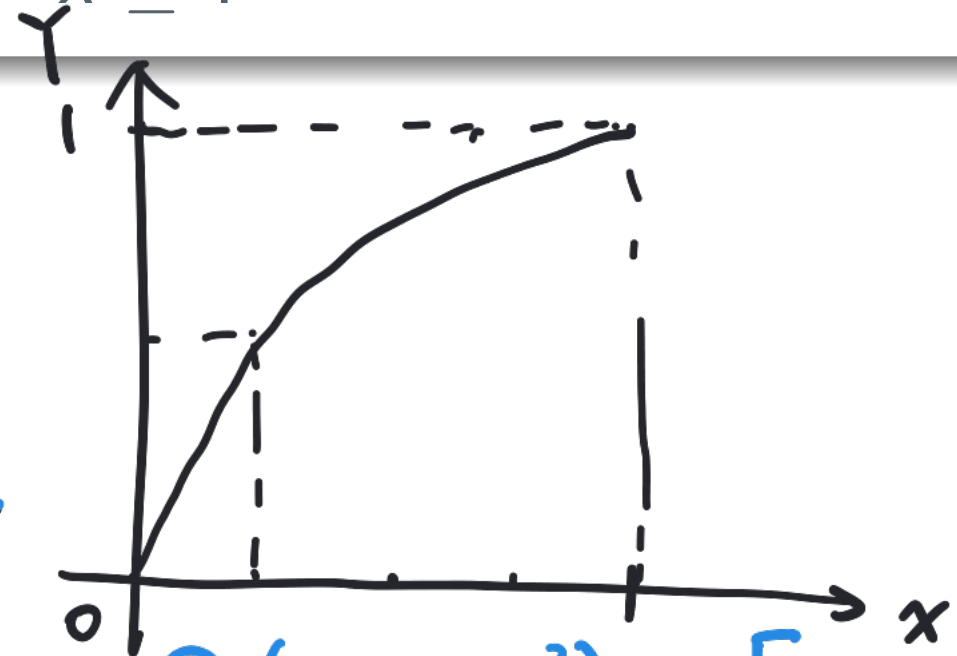
②  $Y = g(X) = \sqrt{X} \Rightarrow B = \{0 < y < 1\}$

③  $F_Y(y) = P(Y \leq y) = P(\sqrt{X} \leq y) = \underbrace{P(X \leq y^2)}_{\uparrow} = \underbrace{F_X(y^2)}$

$$= \begin{cases} 0 \\ y^2 \\ 1 \end{cases}$$

$$\begin{aligned} y &\leq 0 \\ 0 &< y < 1 \\ y &\geq 1 \end{aligned}$$

④  $f_Y(y) = 2y, \quad 0 < y < 1$



# Example 9



Let the random variable  $X$  be uniformly distributed between 0 and 1. Find the probability density function of  $Y = g(X) = \arcsin(X)$ .

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

①  $A = \{0 < x < 1\}$

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

④  $f_Y(y) = \cos y \quad 0 < y < \frac{\pi}{2}$

②  $Y = g(X) = \arcsin(X) \quad B = \{0 < y < \pi/2\}$

③  $F_Y(y) = P(Y \leq y) = P(\arcsin(X) \leq y) = \frac{P(X \leq \sin(y))}{y \leq 0}$   
 $= F_X(\sin y) = \begin{cases} 0 & y \leq 0 \\ \sin y & 0 < y < \pi/2 \\ 1 & y \geq \pi/2 \end{cases}$

# Example 10



Let the random variable  $X$  have the probability density function

$$f_X(x) = \exp(-x) = e^{-x}, \quad x > 0.$$

Find the cumulative distribution function of  $Y = g(X) = X^2$ .

①  $\mathcal{A} = \{x > 0\}$

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x e^{-x} dx & x > 0 \end{cases} = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-x} & x > 0 \end{cases}$$

②  $Y = g(X) = X^2 \Rightarrow \mathcal{B} = \{y > 0\}$

$$\begin{aligned} \text{③ } F_Y(y) &= P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = F_X(\sqrt{y}) \\ &= \begin{cases} 0 & y \leq 0 \\ 1 - e^{-\sqrt{y}} & y > 0 \end{cases} \end{aligned}$$

# Thank You



THANK YOU!