

MATH 451/551

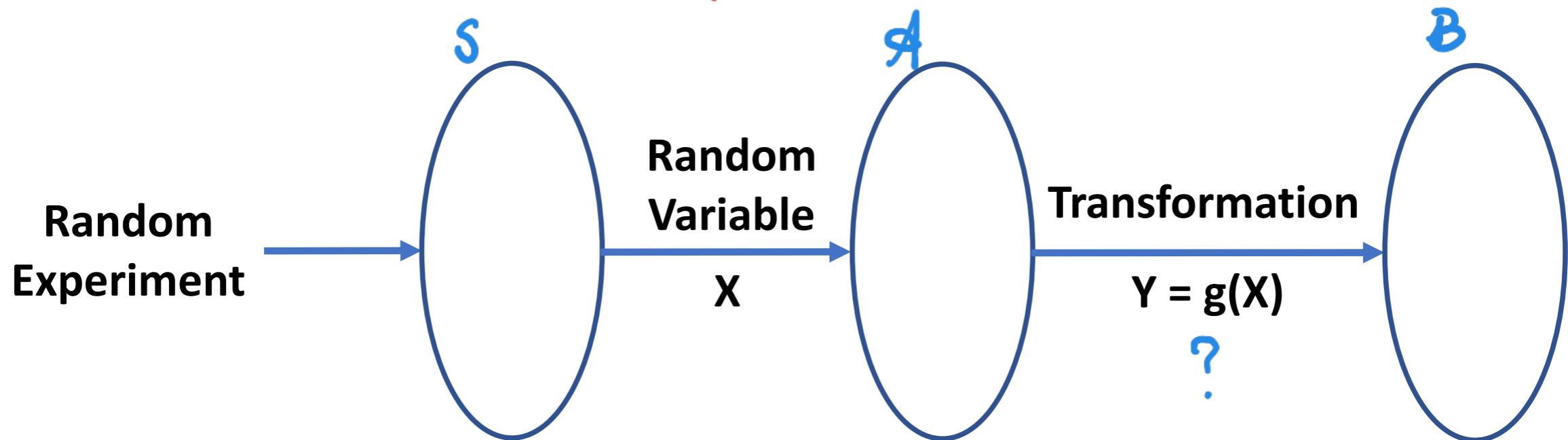
Chapter 3. Random Variables  
3.3 Transformations

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# Transformations of Random Variables

$$F_Y(y) = P(Y \leq y) = P\{g(X) \leq y\} = P\{X \leq g^{-1}(y)\}$$



- ▶  $F_X(x)$ : cumulative distribution function for a random variable  $X$ ;
- ▶  $F_Y(y)$ : cumulative distribution function for a random variable  $Y$ ;
- ▶  $f_X(x)$ : probability mass/density function for a random variable  $X$ ;
- ▶  $f_Y(y)$ : probability mass/density function for a random variable  $Y$ .

$$F_Y(y) = P(Y \leq y) = P\{g(X) \leq y\}.$$



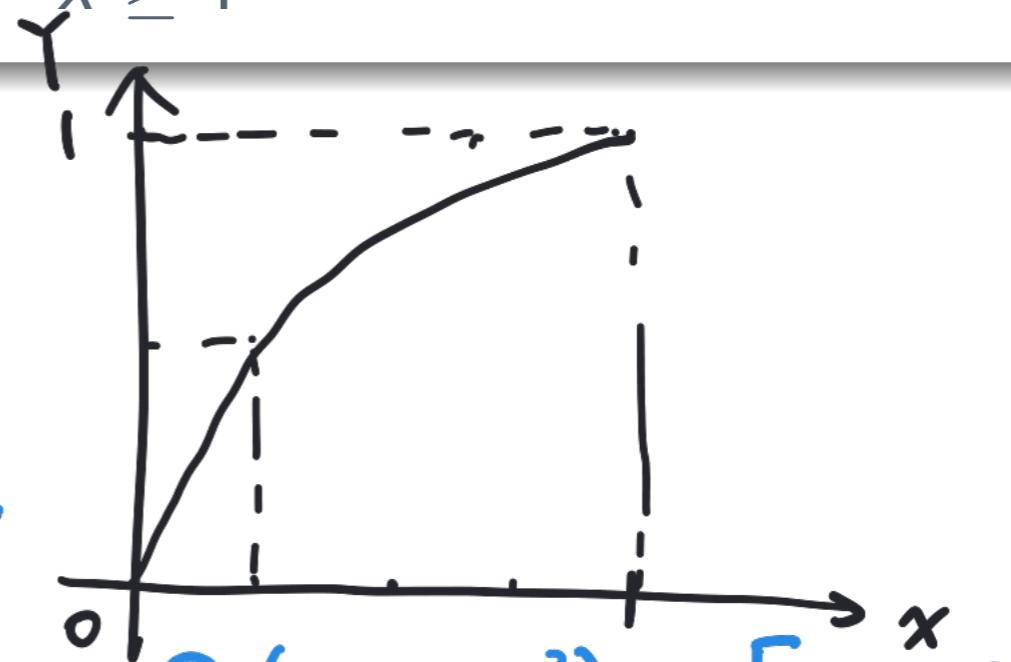
- ▶ We are often interested in finding the distribution of a function of a random variable.
- ▶ Our goal here is to find the cumulative distribution function of  $Y$ .

# Example 8



Let the random variable  $X$  be uniformly distributed between 0 and 1. Find the cumulative distribution function of  $Y = g(X) = \sqrt{X}$ .

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$$



$$\textcircled{1} \quad \mathcal{A} = \{0 < x < 1\}$$

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

$$\textcircled{2} \quad Y = g(X) = \sqrt{X} \Rightarrow \mathcal{B} = \{0 < y < 1\}$$

$$\textcircled{3} \quad F_Y(y) = P(Y \leq y) = P(\sqrt{X} \leq y) = P(X \leq y^2) = F_X(y^2)$$

$$= \begin{cases} 0 & y \leq 0 \\ y^2 & 0 < y < 1 \\ 1 & y \geq 1 \end{cases}$$

$$\textcircled{4} \quad f_Y(y) = 2y, \quad 0 < y < 1$$

# Example 9



Let the random variable  $X$  be uniformly distributed between 0 and 1. Find the probability density function of  $Y = g(X) = \arcsin(X)$ .

$$F_X(x) = \begin{cases} 0, & x \leq 0 \\ x, & 0 < x < 1 \\ 1, & x \geq 1 \end{cases}$$

$$\textcircled{1} \quad A = \{0 < x < 1\}$$

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ x & 0 < x < 1 \\ 1 & x \geq 1 \end{cases}$$

$$\textcircled{4} \quad f_Y(y) = \cos y \quad 0 < y < \frac{\pi}{2}$$

$$\textcircled{2} \quad Y = g(X) = \arcsin(X) \quad B = \{0 < y < \frac{\pi}{2}\}$$

$$\textcircled{3} \quad F_Y(y) = P(Y \leq y) = P(\arcsin(X) \leq y) = P(X \leq \sin(y))$$

$$= F_X(\sin y) = \begin{cases} 0 & y \leq 0 \\ \sin y & 0 < y < \frac{\pi}{2} \\ 1 & y \geq \frac{\pi}{2} \end{cases}$$

# Example 10



Let the random variable  $X$  have the probability density function

$$f_X(x) = \exp(-x) = e^{-x}, \quad x \geq 0.$$

Find the cumulative distribution function of  $Y = g(X) = X^2$ .

①  $\mathcal{A} = \{x > 0\}$

$$F_X(x) = \begin{cases} 0 & x \leq 0 \\ \int_0^x e^{-t} dt & x > 0 \end{cases} = \begin{cases} 0 & x \leq 0 \\ 1 - e^{-x} & x > 0 \end{cases}$$

②  $Y = g(x) = x^2 \Rightarrow \mathcal{B} = \{y > 0\}$

③  $F_Y(y) = P(Y \leq y) = P(X^2 \leq y) = P(X \leq \sqrt{y}) = F_X(\sqrt{y})$

$$= \begin{cases} 0 & y \leq 0 \\ 1 - e^{-\sqrt{y}} & y > 0 \end{cases}$$

Thank You



**THANK YOU!**

