Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 3. Random Variables

3.2 Coontinuous Random Variables

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Motivating Example

Example 1

What probability distribution formalizes the notion of "equally-likely" outcomes in unit interval $[0, 1]$?

$n+1$ values in $[0,1]$

$x = \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \cdots, \frac{n}{n}$ $(n+1)$

$f(x) = \frac{1}{n+1}$

$n \to \infty \Rightarrow \mathcal{A} = \{x \mid 0 \leq x \leq 1\}$

$P(X = \frac{1}{3}) \to 0$

$f(x)$ $P(X = x)$

$n = 100$ $x = 0, 0.01, 0.02, \cdots, 1$

$f(x) = \frac{1}{101}$

$n = 1000$ $x = 0, 0.001, 0.002, \cdots, 1$

$f(x) = \frac{1}{1001}$

? $P(0 < X < \frac{1}{3})$

$P(\frac{1}{9} < X < \frac{4}{9})$

discrete R.V.
pmf
continuous R.V
pdf

Continuous Random Variable

A **continuous random variable** X has a support set $\mathcal{A}$ that is uncountable.

Probability Density Function (PDF)

Probability density function (pdf) existence conditions:

- $0 \leq f(x),\ x \in \mathcal{A}$.
- $\int_{\mathcal{A}} f(x)dx = 1$.
- For $A \subset \mathcal{A}$, $P(X \in A) = \int_A f(x)dx$.
- For real constants $a < b$, $P(a < X < b) = \int_a^b f(x)dx$.
- Endpoints don't matter:

$$P(a < X < b) = P(a \leq X < b) = P(a < X \leq b) = P(a \leq X \leq b),$$

and for any value a, $P(X = a) = 0$.

$P(X=a) = P(X=b) = 0$

Example 2

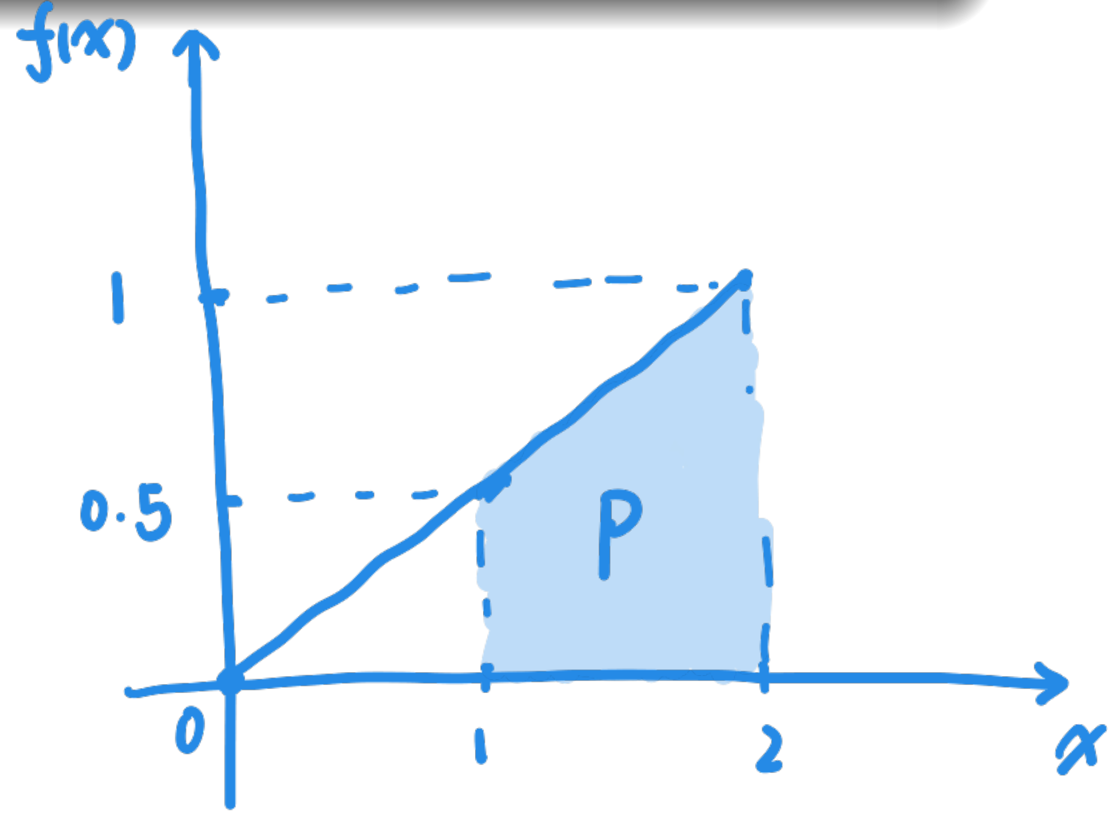
Let the continuous random variable X have pdf

$$f(x) = \frac{x}{2}, \qquad 0 < x < 2.$$

Find the probability that X is greater than 1.

$$P(X>1) = \int_1^2 f(x)\,dx$$

$$= \int_1^2 \frac{x}{2}\,dx = \frac{x^2}{4}\Big|_1^2 = \frac{3}{4}$$



Example

$\lfloor X \rfloor \quad \lfloor 1.4 \rfloor = 1$

$\lfloor 1.98 \rfloor = 1$

round

round $(1.4) = 1$

round $(1.98) = 2$

$\lceil X \rceil \quad \lceil 1.4 \rceil = 2$

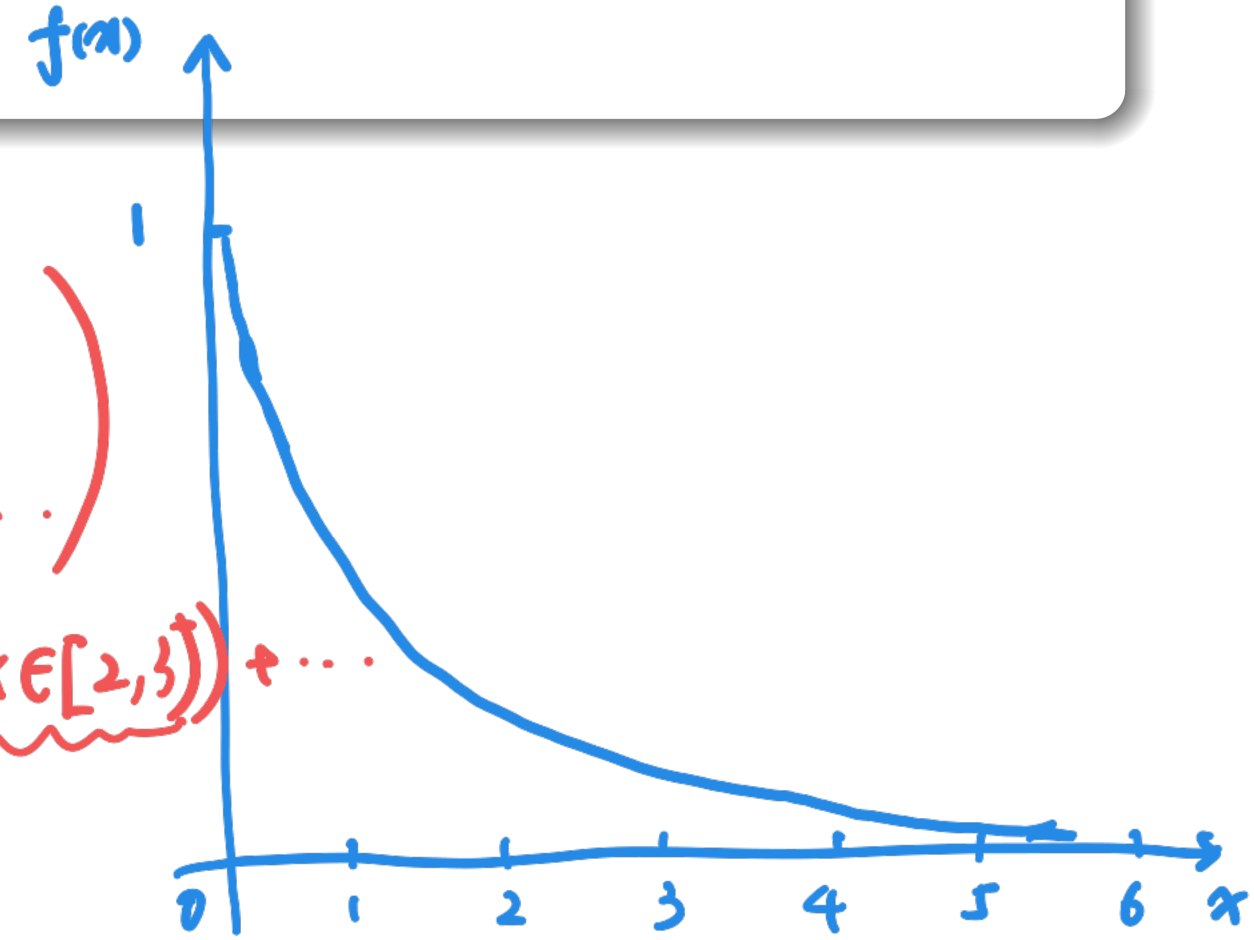
$\lceil 1.98 \rceil = 2$

Example 3

Let the continuous random variable X have probability density function

$$f(x) = e^{-x}, \qquad x > 0$$

Find the probability that $\lfloor x \rfloor$ is even. The floor of X is even is equivalent to X falling in one of these intervals $[0,1),\ [2,3),\ [4,5),\ \cdots$.



$[0,1) \xrightarrow{\lfloor\ \rfloor} 0$

$[1,2) \xrightarrow{\lfloor\ \rfloor} 1$

$[2,3) \xrightarrow{\lfloor\ \rfloor} 2$

$[3,4) \xrightarrow{\lfloor\ \rfloor} 3$

$[4,5) \xrightarrow{\lfloor\ \rfloor} 4$

$\vdots$

$P(\lfloor X \rfloor \text{ is even})$

$= P\big(X \in [0,1) \text{ OR } [2,3) \text{ OR } [4,5) \text{ OR } \cdots\big)$

$= P(X \in [0,1)) + P(X \in [2,3)) + \cdots$

$$P(X\in[0,1)) = \int_0^1 f(x)\,dx = \int_0^1 e^{-x}\,dx = -e^{-x}\Big|_0^1 = 1-e^{-1}$$

$$P(X\in[2,3)) = \int_2^3 f(x)\,dx = \int_2^3 e^{-x}\,dx = -e^{-x}\Big|_2^3 = e^{-2}-e^{-3}$$

$$P(X\in[4,5)) = e^{-4}-e^{-5}$$

$$P(\lfloor X\rfloor \text{ is even}) = (1-e^{-1}) + (e^{-2}-e^{-3}) + (e^{-4}-e^{-5}) + \cdots$$

$$= (1+e^{-2}+e^{-4}+\cdots) - (e^{-1}+e^{-3}+e^{-5}+\cdots)$$

$$= \frac{1(1-\cancel{e^{-2n}})}{1-e^{-2}} - \frac{e^{-1}(1-\cancel{e^{-2n}})}{1-e^{-2}}$$

$$= \frac{1-e^{-1}}{1-e^{-2}} = \frac{1}{1+e^{-1}} \approx 0.7311$$

THANK YOU!