Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 3. Random Variables

3.1 Discrete Random Variables

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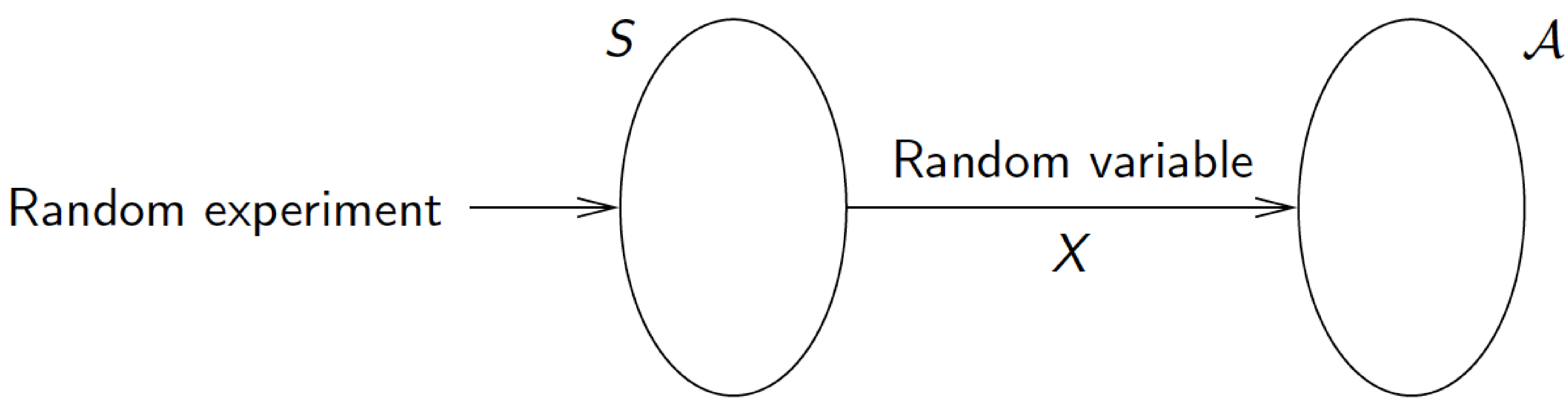
Random Variables

Given a random experiment with an associated sample space S, a **random variable** is a function X that assigns to each element $s \in S$ one and only one real number $X(s) = x$. The **support** of X is the set of real numbers $\mathcal{A} = \{x \mid x = X(s),\ s \in S\}$.

A fair coin is tossed twice. Find the probability that both tosses come up heads.



Discrete Random Variable

Discrete Random Variable

Let X denote a random variable with one-dimensional support $\mathcal{A}$, then a random variable X is discrete if the suppoort $\mathcal{A}$ is countable, that is

- X is a discrete random variable if $\mathcal{A}$ is a finite set, or
- X is a discrete random variable if $\mathcal{A}$ is a denumerable set.

Probability Mass Function

- The probability that X takes on the value x, $P(X = x)$, is defined as the sum of the probabilities of all sample points in S that are assigned the value x. Sometimes, we denote $P(X = x) = p(x)$.
- The probability distribution for a discrete random variable (r.v.) X can represented by a formula, a table, or a graph which provides $P(X = x) = p(x)$ for all x, such that
  - $0 \leq p(x) \leq 1, \ \forall x$
  - $\sum_x p(x) = 1$, where Σ_x is the sum over all possible values, x, oof the r.v. x

Example

Example 1

Toss a fair coin twice times. Find the probability distribution for the number of heads.

probability mass fn (pmf).

discrete r.v.

① Table.
② Figure
③. $f(x)$ (fn).

Example 2

A 5-card hand is dealt from a well-shuffled ~~from~~ a 52-card deck. Let the random variable X be the number of jacks in the hand. Find the support of X and the probability mass function of X.

X = # jacks

$S = \{0, 1, 2, 3, 4\}$

$f(x) = P(X = x)$

value r.v. value

$f(0) = \dfrac{\binom{48}{5}}{\binom{52}{5}}$

$f(1) = \dfrac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}}$

$f(2) = \dfrac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}$

$f(3) = \dfrac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}$

$f(4) = \dfrac{\binom{48}{1}}{\binom{52}{5}}$

♠ 2. 3. ⋯, 10. J. Q. Q. A
♡ 2. 3. ⋯ 10. J Q K A
♣ 2. 3, ⋯ 10 J Q K A
◇ 2. 3, ⋯ 10 J Q K A

2. 3. 4. 5. J

♠J. ♣J 2. 3. 4

♣J. ♡J ♣J 2. 3

♠J ♡J ♣J ◇J . 2

x	0	1	2	3	4
$f(x)$	$\dfrac{\binom{48}{5}}{\binom{52}{5}}$	$\dfrac{\binom{4}{1}\binom{48}{4}}{\binom{52}{5}}$	$\dfrac{\binom{4}{2}\binom{48}{3}}{\binom{52}{5}}$	$\dfrac{\binom{4}{3}\binom{48}{2}}{\binom{52}{5}}$	$\dfrac{\binom{48}{1}}{\binom{52}{5}}$

support

pmf

$$f(x) = \frac{\binom{4}{x}\binom{48}{5-x}}{\binom{52}{5}}, \quad x = 0, 1, 2, 3, 4.$$

Example 3

A box contains 6 green balls and 10 red balls. If we draw 2 balls without replacemennt from the box, let $X = \#$ of green balls drawn, find the probability mass function of X.

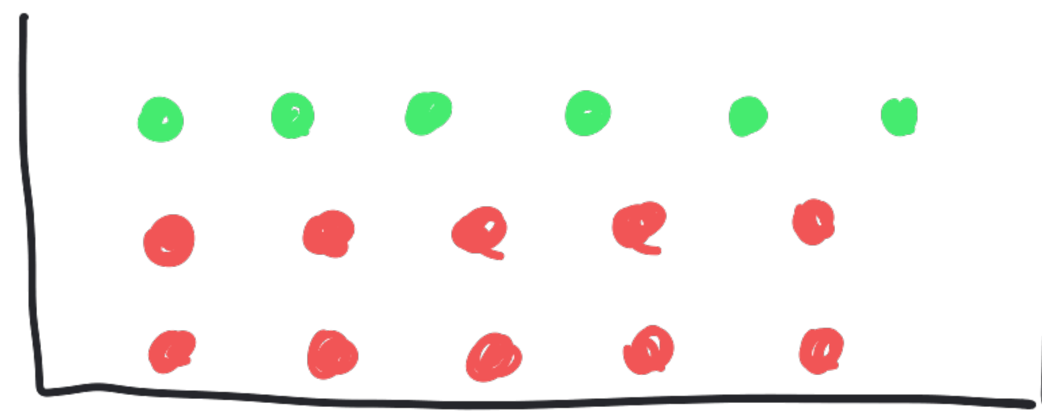
① support.

$\mathcal{X} = \{0, 1, 2\}$

$f(x)$: $f(0) = \dfrac{\binom{10}{2}}{\binom{16}{2}}$

$f(1) = \dfrac{\binom{6}{1}\binom{10}{1}}{\binom{16}{2}}$

$f(2) = \dfrac{\binom{6}{2}}{\binom{16}{2}}$



x	0	1	2
$f(x)$	$\binom{10}{2}/\binom{16}{2}$	$\binom{6}{1}\binom{10}{1}/\binom{16}{2}$	$\binom{6}{2}/\binom{16}{2}$

$f(x) = \dfrac{\binom{6}{x}\binom{10}{2-x}}{\binom{16}{2}}$

$x = 0, 1, 2,$

Example 4

A spinner yields three equally-likely outcomes: 1, 2, 3. If the random variable X denotes the product of the outcomes of two spins, find the probability mass function $f(x)$, $P(X = 6)$, and $P(X \leq 6)$.

$\mathcal{A} = \{1, 2, 3, 4, 6, 9\}$

x	1	2	3	4	6	9
$f(x)$	1/9	2/9	2/9	1/9	2/9	1/9

$$f(x) = \begin{cases} \frac{1}{9} & x = 1, 4, 9 \\ \frac{2}{9} & x = 2, 3, 6 \end{cases}$$

2nd spin

1st spin	1	2	3
1	1	2	3
2	2	4	6
3	3	6	9

Example 5

Flip a fair coin repeatedly until a head appears. Let X be the number of flips required. Find the probability mass function of X and find $P(X \geq 4)$.

$$\mathcal{X} = \{\overset{H}{1}, \overset{TH}{2}, \overset{TTH}{3}, \ldots\}$$

$f(x)$:

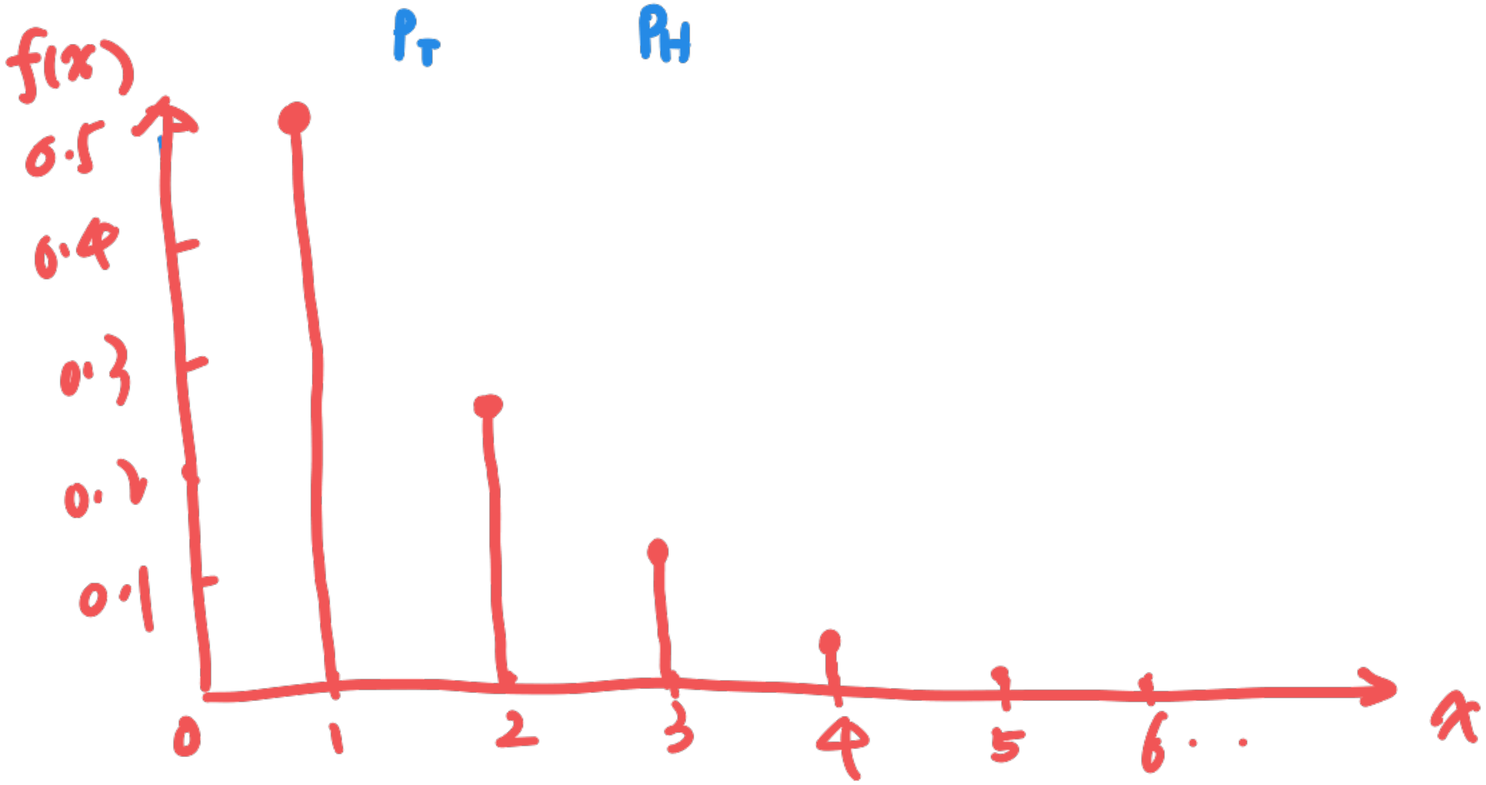
$$f(1) = \frac{1}{2}$$

$$f(2) = \underset{P_T}{\left(\frac{1}{2}\right)} \underset{P_H}{\left(\frac{1}{2}\right)}$$

$$f(3) = \underset{P_T}{\left(\frac{1}{2}\right)^2} \underset{P_H}{\left(\frac{1}{2}\right)}$$

$\vdots$

$$f(x) = \underset{P_T}{\left(\frac{1}{2}\right)^{x-1}} \underset{P_H}{\left(\frac{1}{2}\right)} = \left(\frac{1}{2}\right)^x$$



$$
\begin{aligned}
P(X \geq 4) &= P(X=4) + P(X=5) + P(X=6) + \cdots \\
&= 1 - P(X<4) \\
&= 1 - P(X \leq 3) \\
&= 1 - \{P(X=1) + P(X=2) + P(X=3)\} \\
&= 1 - \left\{\frac{1}{2} + \frac{1}{4} + \frac{1}{8}\right\} \\
&= \frac{1}{8}
\end{aligned}
$$

Arithmetic Seq:

$$\begin{array}{cccccc} a_1 & a_2 & a_3 & a_4 & \cdots & a_n \\ a_1 & a_1+d & a_1+2d & a_1+3d & \cdots & a_1+(n-1)d \end{array}$$

$$S_n = \sum_{i=1}^{n} a_i = \frac{(a_1+a_n)*n}{2}$$

Geometric Seq:

$$\begin{array}{ccccc} a_1 & a_2 & a_3 & \cdots & a_n \\ a_1 & a_1q & a_1q^2, & \cdots & a_1q^{n-1} \end{array}$$

$$S_n = \sum_{i=1}^{n} a_i = \frac{a_1(1-q^n)}{1-q}$$

$f(x) = \left(\frac{1}{2}\right)^x, \quad x = 1, 2, 3, 4, 5, \cdots$

$a_1 = \left(\frac{1}{2}\right)^4 \quad q = \frac{1}{2}$

$$P(X \geq 4) = P(4) + P(5) + P(6) + \cdots$$

$$= \left(\frac{1}{2}\right)^4 + \left(\frac{1}{2}\right)^5 + \left(\frac{1}{2}\right)^6 + \cdots$$

$$= \frac{\left(\frac{1}{2}\right)^4\left(1-\left(\frac{1}{2}\right)^\infty\right)}{1-\frac{1}{2}} = \frac{\left(\frac{1}{2}\right)^4}{\frac{1}{2}} = \left(\frac{1}{2}\right)^3 = \frac{1}{8}$$

THANK YOU!