Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 2. Probability

2.6 Independent Events

$P(A|B) = P(A)$

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Independent Events

Events A and B are **independent** if and only if

$$P(A \cap B) = P(A)P(B).$$

$P(A|B) = P(A)$

$P(B|A) = P(B)$

The following four statements are equivalent:

1. ▶ A and B are independent events,
2. ▶ $P(A \cap B) = P(A)P(B)$,
3. ▶ $P(A|B) = P(A)$,
4. ▶ $P(B|A) = P(B)$.

$P(A|B) = \frac{P(A \cap B)}{P(B)}$

$P(A)$ //

Remarks: The last two statements capture the essence of the independence of two events: the occurrence (or nonoccurrence) of one event doesn't affect the probability of another event occurring. Events that are not independent are said to be **dependent**.

Example 1

A single card is drawn at random from a 52-card deck. Let the event *H* be that the suit of the card is hearts. Let the event *Q* be that the rank of the card is a queen. Are the event *H* and *Q* independent?

4 suits   13 cards.

$P(H) = \frac{13}{52} = \frac{1}{4}$

$P(Q) = \frac{4}{52} = \frac{1}{13}$

$P(H \cap Q) = \frac{1}{52}$

$P(H) * P(Q) = \frac{1}{4} * \frac{1}{13} = \frac{1}{52}$

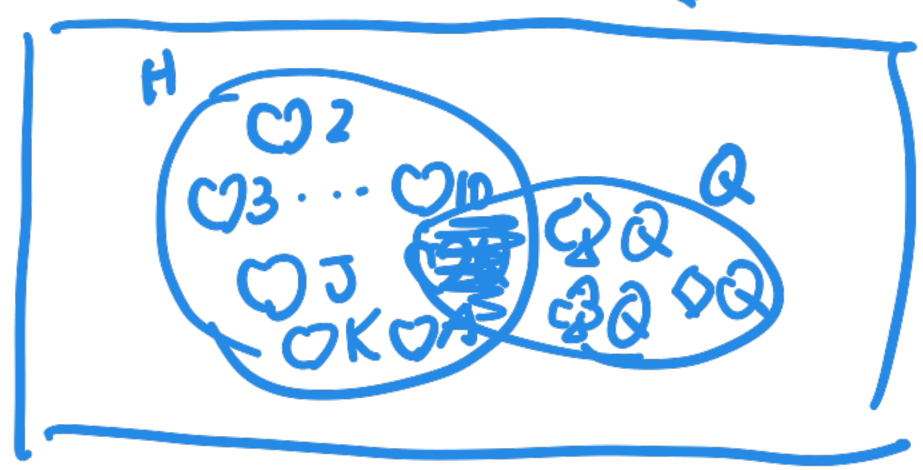
$\therefore$ H & Q are independent

♠ 2, 3, ···, 10, J, Q, K, A

♡ 2, 3, ···, 10, J, Q, K, A

♣ 2, 3, ···, 10, J, Q, K, A

◇ 2, 3, ···, 10, J, Q, K, A



A. B ① A & B disjoint.

② . A & B independent

If A & B disjoint, then A & B independent ?

$A \cap B = \Phi$

$P(A \cap B) = 0$

Assumption

$P(A), P(B) \neq 0$

$$\left.\begin{array}{l} P(A|B) \\ P(B|A) \end{array}\right\} = 0 \neq \begin{array}{l} P(A) \\ P(B) \end{array}$$

Example 2

A fair coin is tossed twice. Show that the events

$S = \{(H,H), (H,T), (T,H), (T,T)\}$

- A: the first toss yields heads, $A = \{(H,H), (H,T)\}$
- B: the second toss yields heads, $B = \{(H,H), (T,H)\}$
- C: the two tosses yield different results, $C = \{(H,T), (T,H)\}$

are pairwise independent, ✓ but $P(A \cap B \cap C) \neq P(A)P(B)P(C)$.

① $P(A \cap B) \stackrel{?}{=} P(A)\,P(B)$

$P(A \cap B) = \frac{1}{4}$

$=$

$P(A) = \frac{2}{4} = \frac{1}{2}$ $P(B) = \frac{2}{4} = \frac{1}{2}$ $P(A)P(B) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

② $P(A \cap C) \stackrel{?}{=} P(A)\,P(C)$

$P(A \cap C) = \frac{1}{4}$

$=$

$P(C) = \frac{2}{4} = \frac{1}{2}$ $P(A)P(C) = \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{4}$

③ $P(B \cap C) = P(B)\,P(C)$

$P(B \cap C) = \frac{1}{4}$

$\parallel$

$P(B)P(C) = \frac{1}{4}$

$P(A \cap B \cap C) = 0$

$\neq$

$P(A)P(B)P(C) = \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} = \frac{1}{8}$

Mutually Independent

Mutually Independent

Events $A_1, A_2, \ldots, A_n$ are **mutually independent** if and only if the probability of occurrence of the intersection of any 2, 3, . . ., or n of these events is equal to the product of their associated probabilities of occurrence.

If the case of three events, A_1, A_2 and A_3, the following equations must be satisfied for three events to be mutually independent:

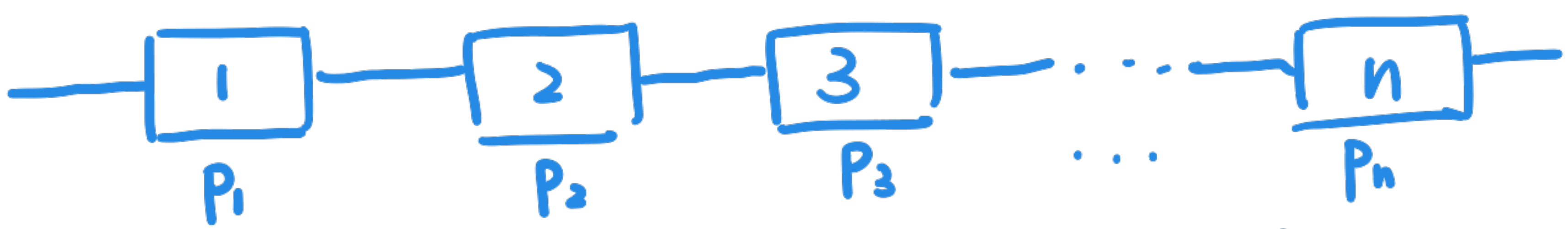
$$
\left.\begin{aligned}
P(A_1 \cap A_2) &= P(A_1)P(A_2) \\
P(A_1 \cap A_3) &= P(A_1)P(A_3) \\
P(A_2 \cap A_3) &= P(A_2)P(A_3)
\end{aligned}\right\} \text{pairwise indep.}
$$

$$
P(A_1 \cap A_2 \cap A_3) = P(A_1)P(A_2)P(A_3)
$$

Example 1

Consider a series system of n mutually independent components with probabilities of functioning $p_1, p_2, \ldots, p_n$. If all components must function for the system to function, find the probability that the system functions.

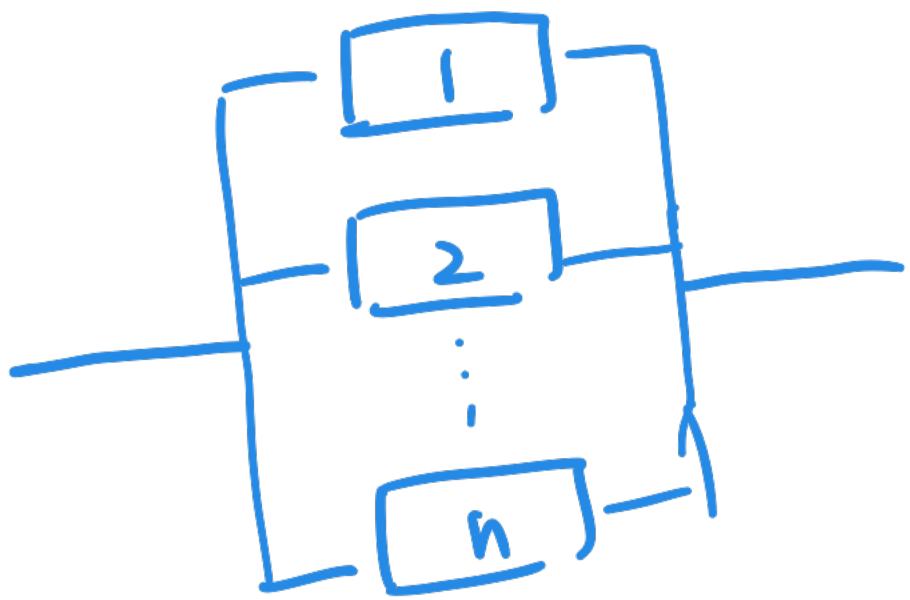
series system



$$P(\text{system funcs}) = P(\text{all components func})$$
$$= P(C_1 \cap C_2 \cap \cdots \cap C_n)$$
$$\overset{\text{mutually}}{\underset{\text{indep}}{=}} P(C_1)\,P(C_2)\cdots P(C_n)$$
$$= p_1 \cdot p_2 \cdot p_3 \cdots p_n$$

Example 2

Consider a parallel system of n mutually independent components with probabilities of function, find the probability that the system functions. $p_1, p_2, \cdots, p_n$



parallel system

$$
\begin{aligned}
P(\text{system func}) &= 1 - P(\text{system fails}) \\
&= 1 - P(\text{all components fail}) \\
&\overset{\text{mutually indep}}{=} 1 - P(C_1 \text{ fails})\, P(C_2 \text{ fails}) \cdots P(C_n \text{ fails}) \\
&= 1 - (1-p_1)(1-p_2)\cdots(1-p_n)
\end{aligned}
$$

Thank You

THANK YOU!