Department of Mathematics
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MATH 451/551

Chapter 2. Probability

2.5 Rule of Bayes

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Rule of Bayes

Let $A_1, A_2, \ldots, A_n$ be a set of events that partition the sample space S, and $P(A_i) > 0$ for $i = 1, 2, \ldots, n$. For any event B with $P(B) > 0$,

$$P(A_j|B) = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{n} P(B|A_i)P(A_i)}$$

$j = 1, 2, \ldots, n$.

Conditional Prob. $P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{P(B|A)P(A)}{P(B)}$ simplest form of rule of Bayes.

$P(A \cap B) = P(B|A)P(A)$

$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)} = \frac{P(B|A_j)P(A_j)}{P(B)} = \frac{P(B|A_j)P(A_j)}{\sum_{i=1}^{n} P(A_i \cap B)} = \frac{P(B|A_j)P(A_j)}{\sum (P(B|A_i)P(A_i))s}$

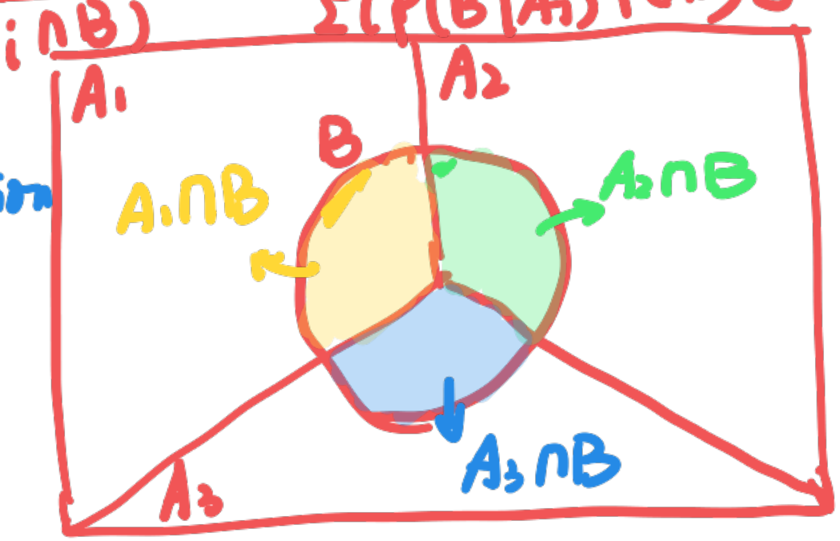
A_1, A_2, A_3 form partition of S

$A_i \cap A_j = \Phi \quad i \neq j$

$\bigcup_{i=1}^{n} A_i = S$

$(A_1 \cap B)$, $(A_2 \cap B)$, $(A_3 \cap B)$ partition of B

$P(B) = \sum_{j=1}^{n} P(A_j \cap B)$



Example 1

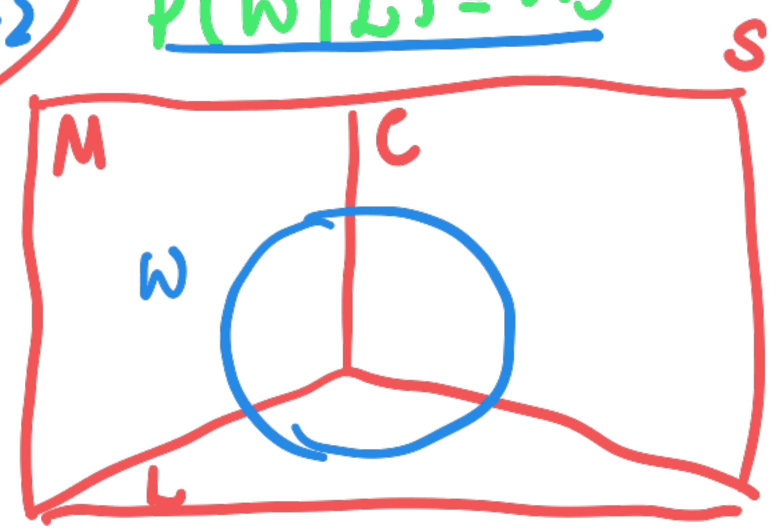
Moe, Curly and Larry are gas station attendants. Moe handles 30% of the customers, Curly handles 50% of the customers, and Larry handles 20% of the customers. They are always supposed to wash the customer's windshield. Moe forgets 1 time in 20, Curly forgets 1 time in 10, and Larry forgets 1 time in 2.

1. What is the probability that a windshield has NOT been washed?
2. Given that a windshield was not washed, what is the probability that it was Curly who didn't wash it?

Let: M = Moe handles the car.
C = Curly handles the car.
L = Larry handles the car
W = the car's windshield is not washed.

$P(M) = 0.3$, $P(C) = 0.5$, $P(L) = 0.2$

$P(W|M) = 0.05$, $P(W|C) = 0.1$, $P(W|L) = 0.5$



① $P(W) = P(M \cap W) + P(C \cap W) + P(L \cap W)$

$= P(W|M)P(M) + P(W|C)P(C) + P(W|L)P(L)$

$= 0.05 * 0.3 + 0.1 * 0.5 + 0.5 * 0.2 = 0.165$

$$P(C|W) = \frac{P(W|C)P(C)}{P(W)} = \frac{0.1 \times 0.5}{0.165} \approx 0.3030$$

Example 2

The “car and goats" problem, also known as the “Monty Hall" paradox or the “Let’s Make a Deal" problem, can be solved using the rule of Bayes. The game show host, Monty Hall, shows you three closed doors. There is a car behind one of the doors and goats behind the other two. If you open the door with the car behind it, you keep the car. You select a door, but before the door is opened, Monty Hall opens one of the other doors to reveal a goat, then gives you the option of switching doors. Is there any advantage to switching?

Let C_1 = car is behind door 1, H_1 = host opens door 1

C_2 = car is behind door 2, H_2 = host opens door 2

C_3 = car is behind door 3, H_3 = host opens door 3.

$P(C_1) = P(C_2) = P(C_3) = \frac{1}{3}$

Assume. pick door 1. opens door 3

$P(H_3 | C_1) = \frac{1}{2}$

$P(H_3 | C_2) = 1$

$P(H_3 | C_3) = 0$

$P(H_3) = P(C_1 \cap H_3) + P(C_2 \cap H_3) + P(C_3 \cap H_3) = P(H_3|C_1)P(C_1) + P(H_3|C_2)P(C_2) + P(H_3|C_3)P(C_3)$

$= \frac{1}{2} \cdot \frac{1}{3} + 1 \cdot \frac{1}{3} + 0 \cdot \frac{1}{3} = \frac{3}{6} = \frac{1}{2}$

$P(C_1 | H_3) = \frac{P(H_3|C_1)P(C_1)}{P(H_3)} = \frac{\frac{1}{2} \cdot \frac{1}{3}}{1/2} = 1/3$

$P(C_2 | H_3) = \frac{P(H_3|C_2)P(C_2)}{P(H_3)} = \frac{1 \times \frac{1}{3}}{1/2} = 2/3$

Independent Events

Events A and B are **independent** if and only if

$$P(A \cap B) = P(A)P(B).$$

The following four statements are equivalent:

- A and B are independent events,
- $P(A \cap B) = P(A)P(B)$,
- $P(A|B) = P(A)$,
- $P(B|A) = P(B)$.

Remarks: The last two statements capture the essence of the independence of two events: the occurrence (or nonoccurrence) of one event doesn't affect the probability of another event occurring. Events that are not independent are said to be **dependent**.

Example 1

A single card is drawn at random from a 52-card deck. Let the event H be that the suit of the card is hearts. Let the event Q be that the rank of the card is a queen. Are the event H and Q independent?

Example 2

A fair coin is tossed twice. Show that the events

- A: the first toss yields heads,
- B: the second toss yields heads,
- C: the two tosses yield different results,

are pairwise independent, but $P(A \cap B \cap C) \neq P(A)P(B)P(C)$.

Thank You

THANK YOU!