Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 2. Probability

2.4 Conditional Probability

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Motivating Examples

- **Meteorology**: What is the probability that it rains tomorrow given that it is raining today?
- **Stock market**: What is the probability that a stock market index rises today given that it dropped yesterday?
- **Genetics**: What is the probability that a child will have blue eyes given that one parent has blue eyes?
- **Economic**: What is the probability that government revenue will increase next month given that there is a specified small increase in unemployment this month?

Motivating Example

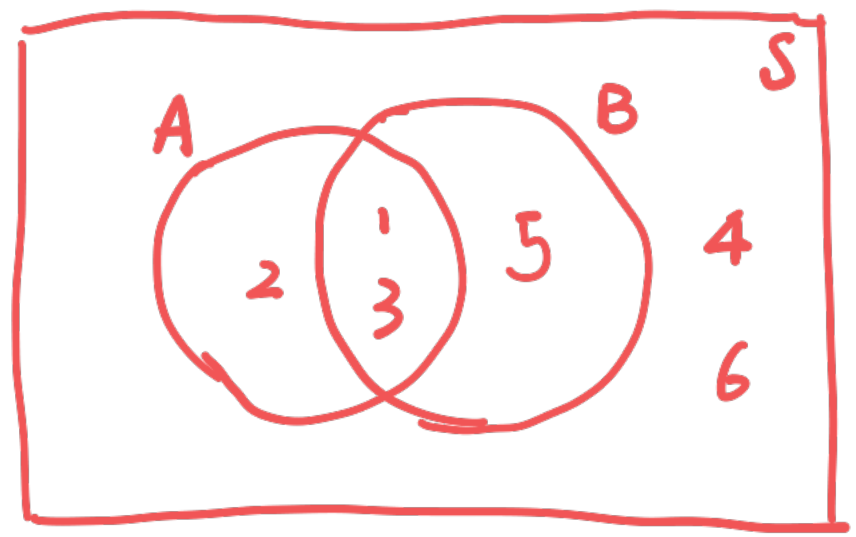
Toss a fair die and observe the number of spots on the up face. Let the event A correspond to tossing a 1, 2, or 3. Let the event B correspond to tossing an odd number. What is the probability of A? What is the probability of A given that B has occurred?

$S = \{1, 2, 3, 4, 5, 6\}$

$A = \{1, 2, 3\}$ $B = \{1, 3, 5\}$

$P(A) = \frac{N(A)}{N(S)} = \frac{3}{6} = \frac{1}{2}$

$P(A \mid B) = \frac{N(A \cap B)}{N(B)} = \frac{2}{3}$



Conditional Probability

Conditional Probability

If A and B are two events in the sample space S associated with a random experiment, then the probability of A given B is

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

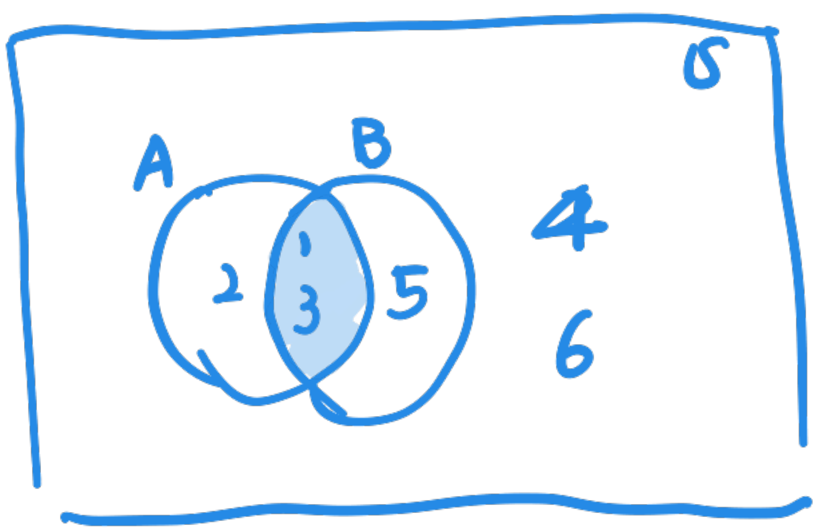
provided $P(B) > 0$.

$$P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{P(A \cap B)}{P(A)}, \quad P(A) > 0.$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{1/3}{1/2} = 2/3$$

$$P(A \cap B) = \frac{2}{6} = 1/3$$

$$P(B) = \frac{3}{6} = \frac{1}{2}$$



Example 1

The results of a random sample of 100 subjects classified by their gender and eye color is given in the following table. If one of the subjects is selected at random

	Blue	Green	Other	
Male	26	23	24	73
Female	13	12	2	

1. find the probability that they have blue eyes given that they are male,
2. find the probability that they are female given that they have green eyes.

let B = blue eyes; G = green eyes

O = other colored eyes

M = male; F = female

$$P(B|M) = \frac{P(B\cap M)}{P(M)} = \frac{26/100}{(26+23+24)/100} = \frac{26}{73}$$

$$P(F|G) = \frac{P(F\cap G)}{P(G)} = \frac{12/100}{(23+12)/100} = \frac{12}{35}$$

Example 2

Consider the events A_1 and A_2 with associated probabilities $P(A_1) = 0.3$, $P(A_2) = 0.5$, and $P(A_1 \cap A_2) = 0.2$. Find $P(A_1|A_2)$ and $P(A_2|A_1)$.

$$P(A_1|A_2) = \frac{P(A_1 \cap A_2)}{P(A_2)} = \frac{0.2}{0.5} = 0.4$$

$$P(A_2|A_1) = \frac{P(A_1 \cap A_2)}{P(A_1)} = \frac{0.2}{0.3} = \frac{2}{3}$$

Example 3

There are 10,000 doctors placed on a low daily dose of aspirin and 10,000 other doctors placed on a placebo for a year. During this year, 107 of those on aspirin have a heart attack, while 187 of those on the placebo have a hear attack. If a doctor from the study is selected at random, what is the probability that the doctor will have been on aspirin and had a heart attack?

A ∩ H

let A = on aspirin
H = had a heart attack

$$P(H|A) = \frac{107}{10,000}$$

$$P(A) = \frac{10000}{10000+10000} = \frac{1}{2}$$

$$P(A \cap H) = P(H|A)P(A) = \frac{107}{10000} \times \frac{1}{2} \approx 0.00535$$

$$P(A|B) = \frac{P(A \cap B)}{P(B)}$$

$$P(A \cap B) = P(A|B)P(B)$$

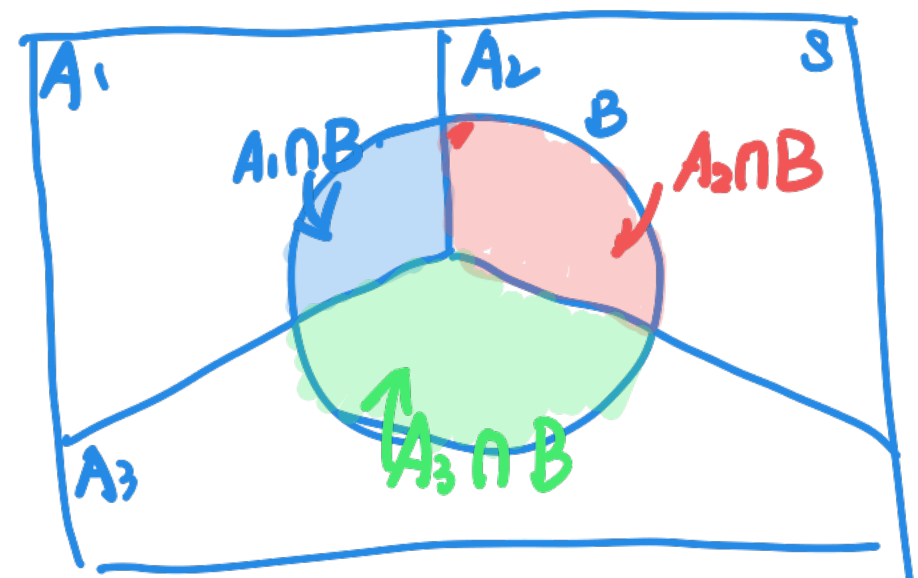
$$= P(B|A)P(A)$$

Law of Total Probability

Let $A_1, A_2, \dots, A_n$ be a set of events that partition the sample space S, and $P(A_i) > 0$ for $i = 1, 2, \dots, n$. For any event B,

$$P(B) = \sum_{i=1}^{n} P(B|A_i)P(A_i)$$

Proof:
$$\begin{aligned}P(B) &= P(B \cap S) \\ &= P\left(B \cap \left(\bigcup_{i=1}^{n} A_i\right)\right) \\ &= P\left(\bigcup_{i=1}^{n} (B \cap A_i)\right) \\ &= \sum_{i=1}^{n} P(B \cap A_i) \\ &= \sum_{i=1}^{n} P(B|A_i)P(A_i)\end{aligned}$$

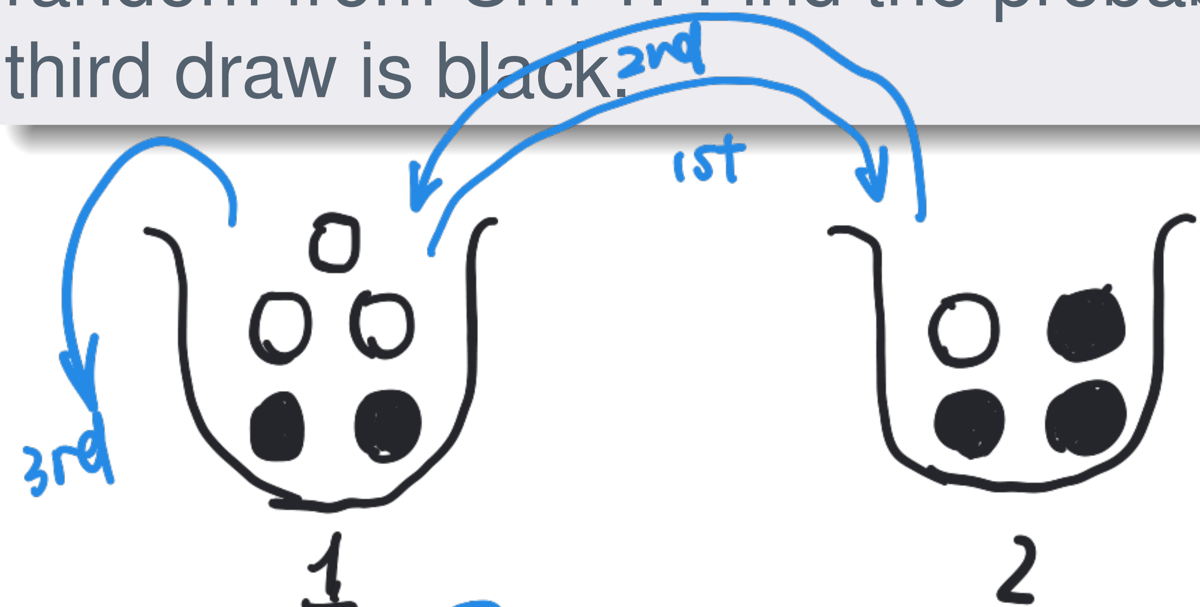


$P(B) = P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)$

$= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3)$

Example 4

Urn 1 contains three white balls and two black balls. Urn 2 contains one white ball and three black balls. A ball is selected at random from Urn 1 and transferred to Urn 2. Next, a ball is selected at random from Urn 2 and transferred to Urn 1. Finally, a ball is selected at random from Urn 1. Find the probability that the ball selected on this third draw is black.



W_i = white ball on draw i, $i = 1, 2, 3$

B_i = black ball on draw i, $i = 1, 2, 3$

$P(B_1) = \frac{2}{5}$ $\quad P(W_1) = \frac{3}{5}$ $\quad$ 1st

2nd. B_1 ↑

$P(B_2 | B_1) = \frac{4}{5}$

$P(W_2 | B_1) = \frac{1}{5}$

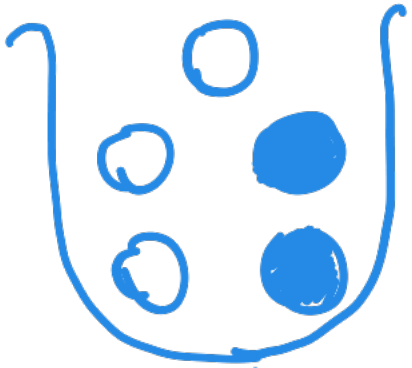
W_1

$P(B_2 | W_1) = \frac{3}{5}$

$P(W_2 | W_1) = \frac{2}{5}$

3rd

$B_1 \cap B_2$



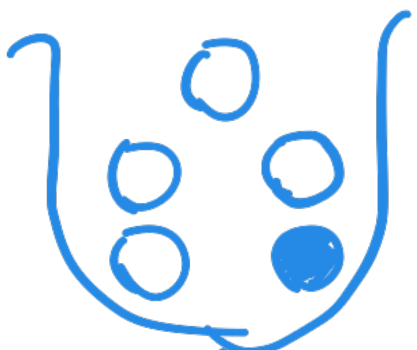
1st

$P(B_3 | (B_1 \cap B_2)) = \frac{2}{5}$

$P(W_3 | (B_1 \cap B_2)) = \frac{3}{5}$

$\frac{2}{5} * \frac{4}{5} * \frac{2}{5}$

$B_1 \cap W_2$



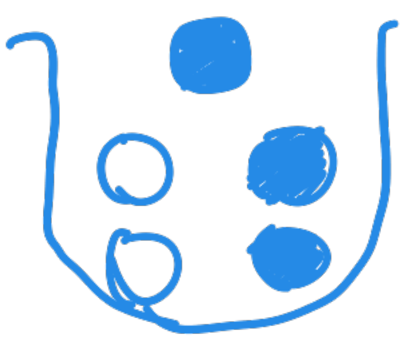
1st

$P(B_3 | (B_1 \cap W_2)) = \frac{1}{5}$

$P(W_3 | (B_1 \cap W_2)) = \frac{4}{5}$

$\frac{2}{5} * \frac{1}{5} * \frac{1}{5}$

$W_1 \cap B_2$



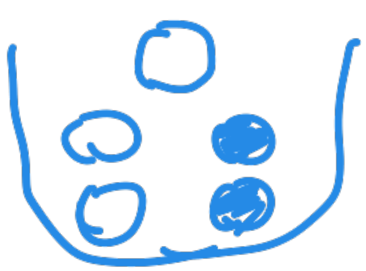
1st

$P(B_3 | (W_1 \cap B_2)) = \frac{3}{5}$

$P(W_3 | (W_1 \cap B_2)) = \frac{2}{5}$

$\frac{3}{5} * \frac{3}{5} \times \frac{3}{5}$

$W_1 \cap W_2$



1st

$P(B_3 | (W_1 \cap W_2)) = \frac{2}{5}$

$P(W_3 | (W_1 \cap W_2)) = \frac{3}{5}$

$\frac{2}{5} * \frac{2}{5} \times \frac{2}{5}$

$$P(B_3) = P(B_3 | B_1 \cap B_2) * P(B_2 | B_1) P(B_1)$$

$$+ P(B_3 | B_1 \cap W_2) * P(W_2 | B_1) P(B_1)$$

$$+ P(B_3 | (W_1 \cap B_2)) * P(B_2 | W_1) P(W_1)$$

$$+ P(B_3 | W_1 \cap W_2) * P(W_2 | W_1) P(W_1)$$

$$= \frac{57}{125} = 0.456$$

Thank you!