Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 2. Probability

2.2 Probability Axioms

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Probability Axioms

Relative Frequency

Perform a random experiment n times. Let x be the number of times that the event A occurs. The ratio x/n is the **relative frequency** of the event A in the n experiment.

Estimate the probability that a coin comes up “heads" on a single toss of a fair coin.

Limiting Relative Frequency

Limiting Relative Frequency

The **limiting relative frequency** of the event A

$$P(A) = \lim_{n\to\infty} \frac{x}{n}$$

is called the

- probability that the outcome of the random experiment is in A, or
- probability of event A,

if the limit exists.

Classical Approach

If the sample space S consists of equally-likely outcomes, then the probability oof event A is the ratio of the number of elements in A to the number of elements in S:

elements

$$P(A) = \frac{N(A)}{N(S)}.$$

Toss a fair cooin three times and observed sequence of heads and tails. Find the probability that all three outcomes are heads.

$S = \{HHH, HHT, HTH, HTT, THH, THT, TTH, TTT\}$ equally-likely ✓

Define event $A = \{$all three outcomes are heads$\} = \{HHH\}$

$$P(A) = \frac{N(A)}{N(S)} = \frac{1}{8}$$

Kolmogorov Axioms

Kolmogorov Axioms

Consider a random experiment with sample space S and an event $A \subset S$ of interest. If $P(A)$ is defined and

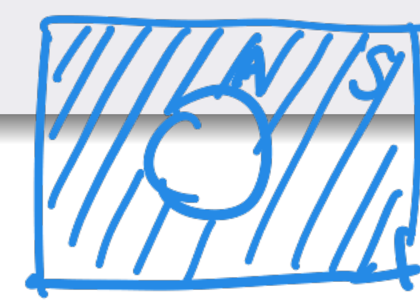
- **Axiom 1.** $P(A) \geq 0$,
- **Axiom 2.** $P(A_1 \cup A_2 \cup \cdots) = P(A_1) + P(A_2) + \cdots$ where $A_1, A_2, \ldots$ are disjoint events, and
- **Axiom 3.** $P(S) = 1$,

then $P(A)$ is the probability of event A occurring.

$P(A)=0 \Rightarrow A$ is impossible.
$P(A)=1 \Rightarrow A$ is certainty.

Complementary Probability

For each $A \subset S$, $P(A) = 1 - P(A^c)$.



$A \cup A^c = S$ $\quad P(A \cup A^c) = P(A) + P(A^c)$

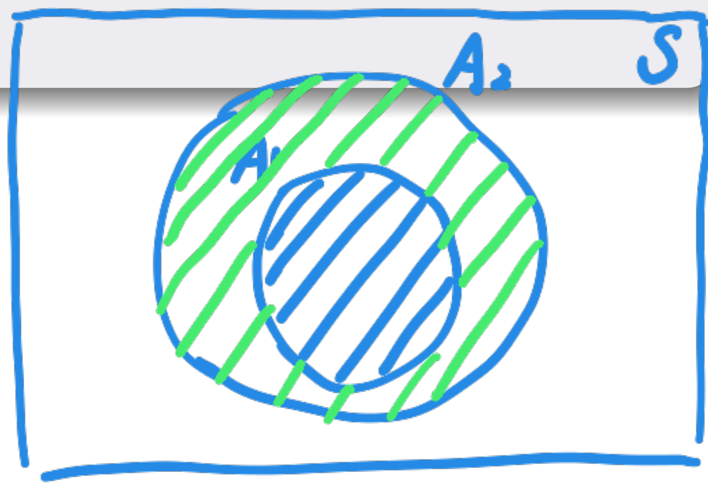
$A \cap A^c = \phi$ $\quad = P(S) = 1$ $\quad \Rightarrow P(A) + P(A^c) = 1$

$\Rightarrow P(A) = 1 - P(A^c)$

Theorems

Theorem 2.2

If $A_1, A_2 \subset S$ such that $A_1 \subset A_2$ then ① $P(A_1^c \cap A_2) = P(A_2) - P(A_1)$, and therefore ② $P(A_1) \leq P(A_2)$.



partition A_2

$A_2 = A_1 \cup (A_2 \cap A_1^c)$

$\underline{\phi = A_1 \cap (A_2 \cap A_1^c)}$

① $P(A_2) = P(A_1 \cup (A_2 \cap A_1^c)) = \underline{P(A_1) + P(A_2 \cap A_1^c)}$

$\Rightarrow P(A_1^c \cap A_2) = P(A_2) - P(A_1)$

② $P(A_2) - P(A_1) = P(\underline{A_1^c \cap A_2}) \geq 0$

$\Rightarrow P(A_1) \leq P(A_2)$

Theorem 2.3

$P(\Phi) = 0.$

let $A = \Phi$ then $A^c = S$

$P(\Phi) = P(A) = 1 - P(A^c) = 1 - \underbrace{P(S)}_{1} = 0$

Theorem 2.4

For every $A \subset S$, $0 \le P(A) \le 1.$

$\Phi \subseteq A \subseteq S$

$P(\Phi) \le P(A) \le P(S)$.

$0 \le P(A) \le 1$

Theorems

Theorem 2.5

(Addition Rule) If $A_1, A_2 \subset S$ then

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2).$$

$A_1 \cup A_2 = A_1 \cup (A_1^c \cap A_2)$

$A_2 = (A_1 \cap A_2) \cup (A_1^c \cap A_2)$

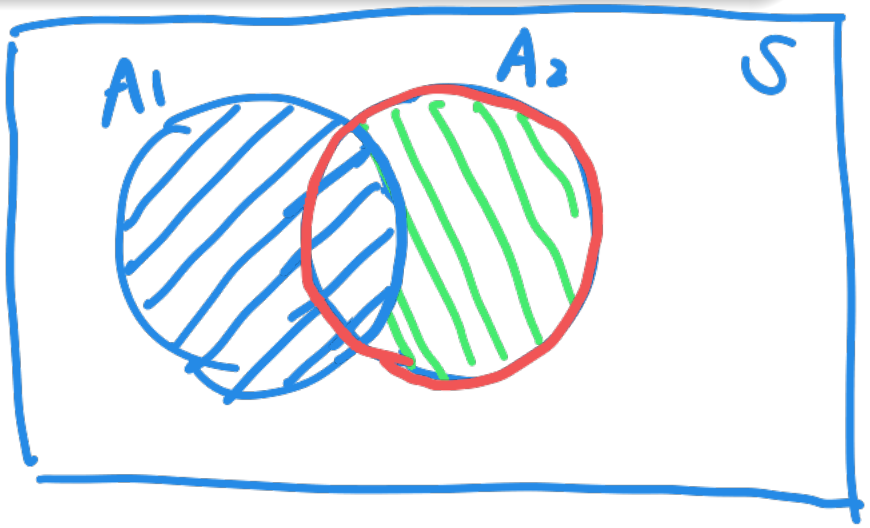
① $P(A_1 \cup A_2) = P(A_1) + P(A_1^c \cap A_2)$

② $P(A_2) = P(A_1 \cap A_2) + P(A_1^c \cap A_2)$

① - ②

$P(A_1 \cup A_2) - P(A_2) = P(A_1) - P(A_1 \cap A_2)$

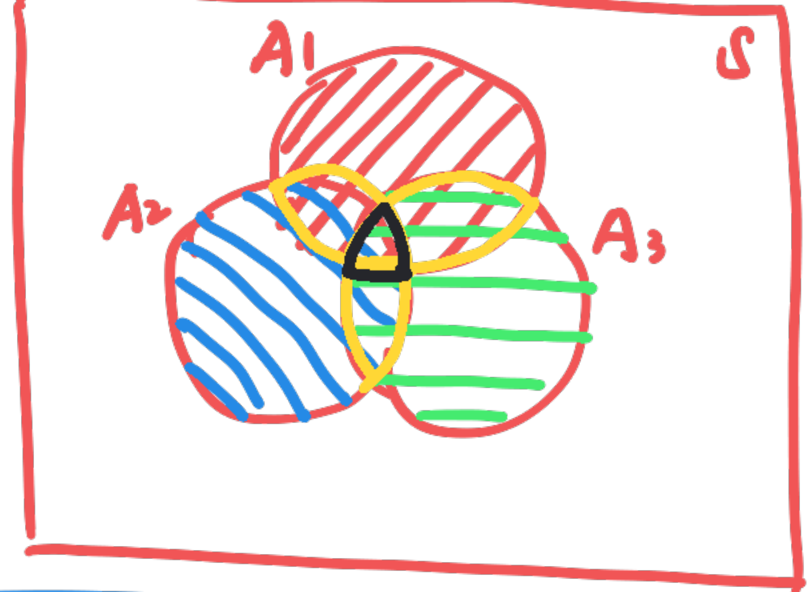
$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$



$P(A_1 \cup A_2 \cup A_3)$

$= P(A_1) + P(A_2) + P(A_3)$

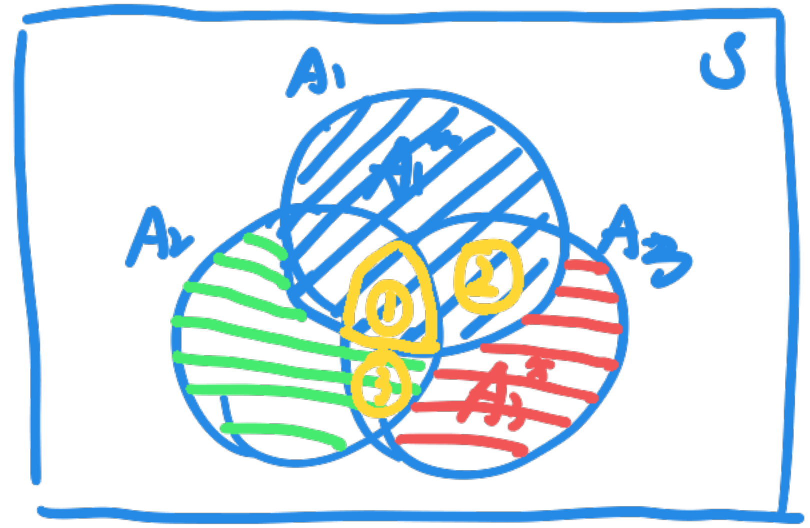
$- P(A_1 \cap A_2) - P(A_1 \cap A_3) - P(A_2 \cap A_3)$ *



$A_1^* + P(A_1 \cap A_2 \cap A_3)$ $\quad A_1^* \cap A_2^* = \phi$

$A_2^* = A_1^c \cap A_2 \qquad (A_1 \cap A_2) \cup A_2^* = A_2$

$A_3^* = (A_1 \cup A_2)^c \cap A_3 = A_1^c \cap A_2^c \cap A_3$



$P(A_2) = P\{(A_1 \cap A_2) \cup (A_1^c \cap A_2)\}$

$= P(A_1 \cap A_2) + P(A_2^*)$

$P(A_3) = P(A_1 \cap A_2 \cap A_3) + P(A_1 \cap A_2^c \cap A_3) + P(A_1^c \cap A_2 \cap A_3) + P(A_3^*)$

$P(A_1 \cup A_2 \cup A_3) = P(A_1^* \cup A_2^* \cup A_3^*) = P(A_1^*) + P(A_2^*) + P(A_3^*)$

$= P(A_1) + P(A_2) - P(A_1 \cap A_2) + P(A_3) - P(A_1 \cap A_2 \cap A_3) - P(A_1 \cap A_2^c \cap A_3)$

$- P(A_1^c \cap A_2 \cap A_3)$

$= P(A_1) + P(A_2) - P(A_1 \cap A_2) + P(A_3) - P(A_1 \cap A_2 \cap A_3)$

$- \{P(A_1 \cap A_3) - P(A_1 \cap A_2 \cap A_3)\} - \{P(A_2 \cap A_3) - P(A_1 \cap A_2 \cap A_3)\}$

Theorems

If events $A_1, A_2, \ldots, A_n$ are disjoint and their union is the sample space S, then they form a $\boxed{\text{partition}}$ of S.

$$A_i \cap A_j = \Phi \text{ for } i \neq j \qquad \text{and} \qquad S = \cup_{i=1}^{n} A_i$$

Theorem 2.6

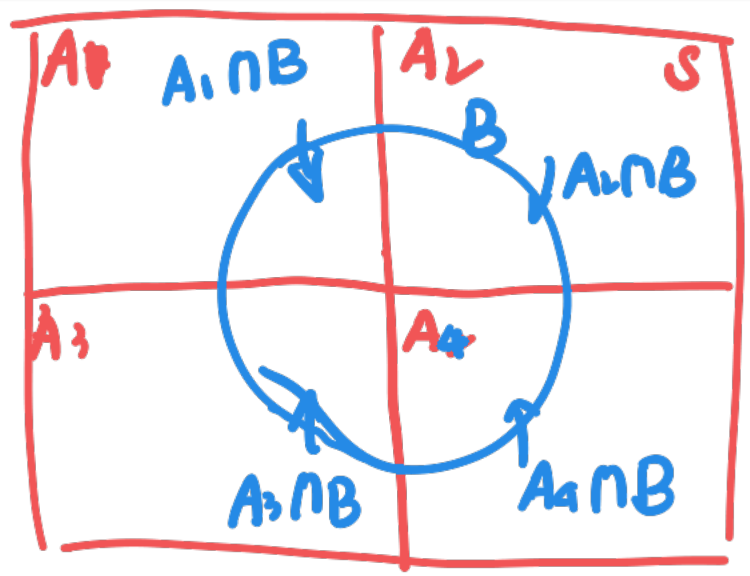
For events $A_1, A_2, \ldots, A_n$ that form a partition of S and another event $B \subset S$,

$$P(B) = \sum_{i=1}^{n} P(B \cap A_i).$$

$A_i \cap A_j = \Phi$, $i \neq j$, $i, j = 1, 2, 3, 4$

$\bigcup_{i=1}^{4} A_i = S$

$P(B) = P((A_1 \cap B) \cup (A_2 \cap B) \cup (A_3 \cap B) \cup (A_4 \cap B)) = \sum_{i=1}^{4} P(A_i \cap B)$



Example 2

Consider the events A_1 and A_2 with associated probabilities $P(A_1) = 0.3$, $P(A_2) = 0.5$, and $P(A_1 \cup A_2) = 0.6$. Find 1 $P(A_1 \cap A_2)$ and 2 $P(A_1 \cap A_2^c)$.

$$P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$$

$$0.6 = 0.3 + 0.5 - P(A_1 \cap A_2) \Rightarrow P(A_1 \cap A_2) = 0.2$$

$$(A_1 \cap A_2) \cap (A_1 \cap A_2^c) = \Phi$$

$$(A_1 \cap A_2) \cup (A_1 \cap A_2^c) = A_1$$

$$P(A_1) = P(A_1 \cap A_2) + P(A_1 \cap A_2^c)$$

$$0.3 = 0.2 + P(A_1 \cap A_2^c) \Rightarrow P(A_1 \cap A_2^c) = 0.1$$

Example 3

Roll a pair of fair dice. Find $P(A_1), P(A_2), \ldots, P(A_5)$ for the events

- A_1: rolling a total of 7
- A_2: rolling a total of 12
- A_3: rolling doubles
- A_4: rolling 5, either individually or as a total
- A_5: rolling numbers that differ by 3

$A_1 = \{(1,6),(6,1),(2,5),(5,2),(3,4),(4,3)\}$

$P(A_1) = \frac{N(A_1)}{N(S)} = \frac{6}{36} = \frac{1}{6}$

$A_2 = \{(6,6)\} \Rightarrow P(A_2) = \frac{1}{36}$

$A_3 = \{(1,1),(2,2),(3,3),(4,4),(5,5),(6,6)\}$

$P(A_3) = \frac{N(A_3)}{N(S)} = \frac{6}{36} = \frac{1}{6}$

$A_4 = \{(1,5),(2,5),(3,5),(4,5),(5,5),(6,5),$
$(5,1),(5,2),(5,3),(5,4),(5,6),$
$(1,4),(4,1)(2,3),(3,2)\}$

$P(A_2) = \frac{15}{36}$

	1	2	3	4	5	6
1	(1,1)	(1,2)				
2	(2,1)					
3						
4						
5						
6						(6,6)

THANK YOU!