Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 1. Introduction

1.3.2 Algebra of Sets

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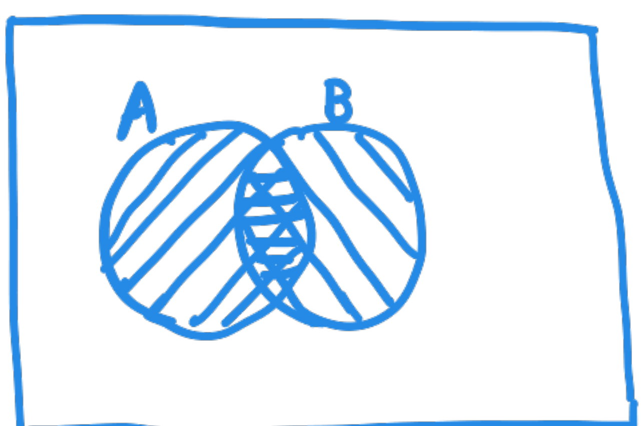
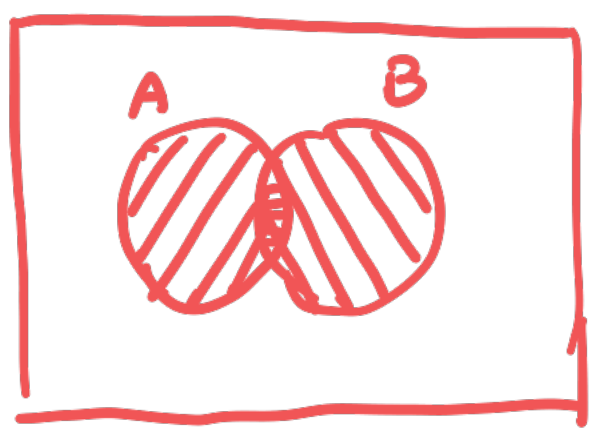
Algebra of Sets

Commutativity

$$
\begin{aligned}
① \quad (A \cup B) &= (B \cup A) \\
?\ ② \quad (A \cap B) &= (B \cap A)
\end{aligned}
$$

$A \cup B$ $\qquad$ $B \cup A$





$A \cup B = B \cup A$

Proof: ① $(A \cup B) \subset (B \cup A)$

$x \in (A \cup B) \Rightarrow x \in A \Rightarrow x \in (B \cup A)$

$x \in B$

$\therefore (A \cup B) \subset (B \cup A)$

② $(B \cup A) \subset (A \cup B)$

$x \in (B \cup A) \Rightarrow x \in B \Rightarrow x \in (A \cup B)$

OR

$x \in A \qquad x \in (A \cup B)$

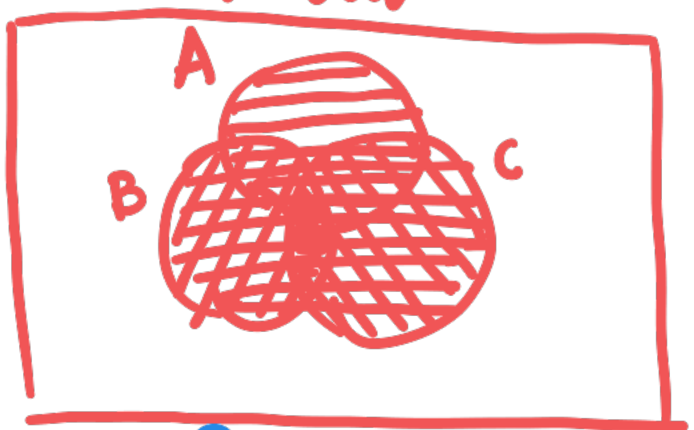
$\therefore (B \cup A) \subset (A \cup B)$

$(A \cup B) = (B \cup A)$

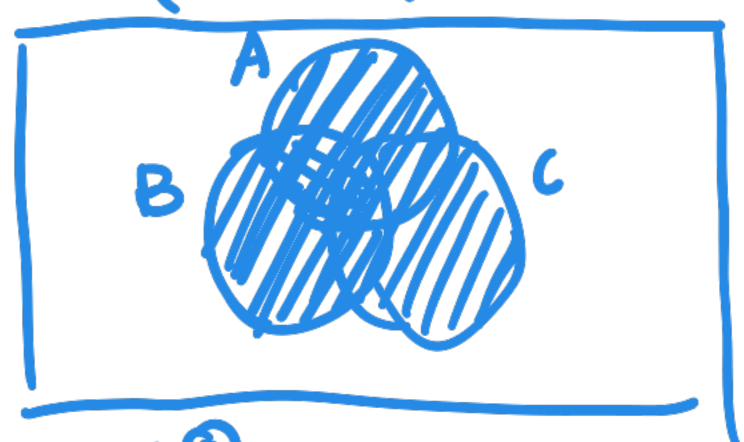
Associativity

$$① \quad A \cup (B \cup C) = (A \cup B) \cup C$$
$$② \quad A \cap (B \cap C) = (A \cap B) \cap C$$

$A \cup (B \cup C)$



$(A \cup B) \cup C$



$A \cup (B \cup C) = (A \cup B) \cup C$

Proof: ① $A \cup (B \cup C) \subseteq (A \cup B) \cup C$

$x \in A \cup (B \cup C) \Rightarrow x \in A \Rightarrow x \in (A \cup B) \Rightarrow x \in (A \cup B) \cup C$

OR

$x \in (B \cup C) \Rightarrow x \in B \Rightarrow x \in (A \cup B) \Rightarrow$

OR $x \in C \Rightarrow x \in (A \cup B) \cup C$

$\therefore x \in (A \cup B) \cup C$

② $(A \cup B) \cup C \subseteq A \cup (B \cup C)$

$x \in (A \cup B) \cup C$

$\Rightarrow x \in (A \cup B) \Rightarrow x \in A \Rightarrow x \in A \cup (B \cup C)$

OR $x \in B \Rightarrow x \in (B \cup C)$

OR

$x \in C \Rightarrow x \in (B \cup C)$

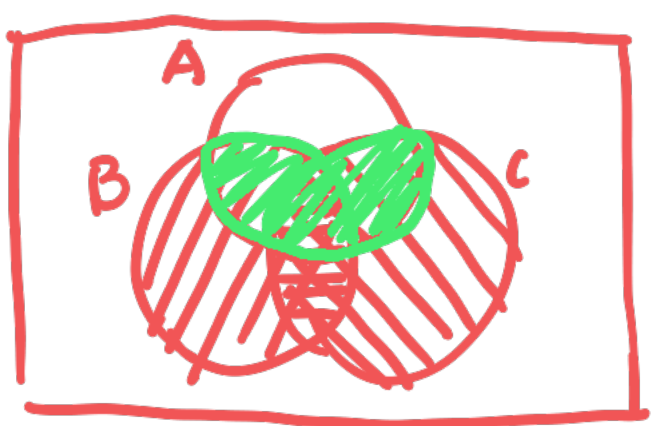
$\therefore x \in A \cup (B \cup C)$

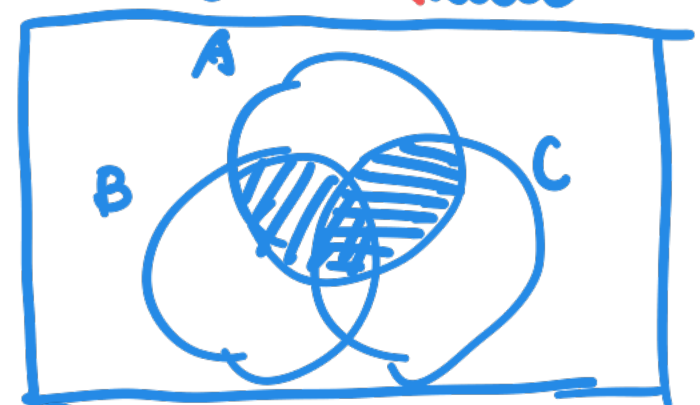
Distribution Laws

① $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$

② $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$

$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$





Proof ① $A \cap (B \cup C) \subseteq (A \cap B) \cup (A \cap C)$

$x \in A \cap (B \cup C)$

$\Rightarrow x \in A$

AND

$x \in (B \cup C) \Rightarrow x \in B$ OR $x \in C$

$\Rightarrow x \in A$ and $x \in B$ OR $x \in A$ and $x \in C$

$\Rightarrow x \in (A \cap B)$ OR $x \in (A \cap C)$

② $(A \cap B) \cup (A \cap C) \subseteq A \cap (B \cup C)$

$x \in (A \cap B) \cup (A \cap C)$

$\Rightarrow x \in A \cap B \Rightarrow \overbrace{x \in A \text{ and } x \in B \Rightarrow x \in (B \cup C)}^{x \in A \cap (B \cup C)}$

OR

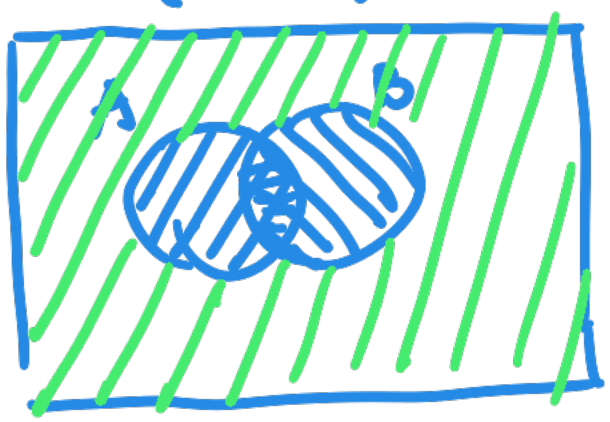
$x \in A \cap C \Rightarrow \underbrace{x \in A \text{ and } x \in C \Rightarrow x \in B \cup C}_{x \in A \cap (B \cup C)}$

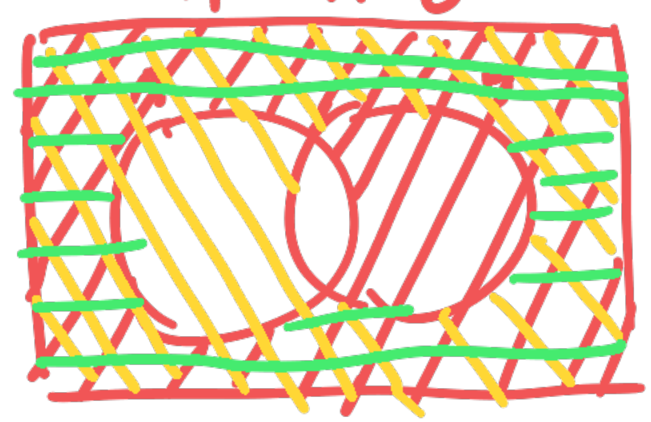
Algebra of Sets

DeMorgans Laws

1. $(A \cup B)^c \quad = \quad (A^c) \cap (B^c)$
2. $(A \cap B)^c \quad = \quad (A^c) \cup (B^c)$

$(A \cup B)^c$ $=$ $A^c \cap B^c$





Power Set

A **power set** associate with a set A is a set that consists of all possible subsets of A. including Φ, A

1. If $A = \{a, b\}$, what is the power set of A?

$n(A) = n$

$\{\Phi, \{a\}, \{b\}, \{a,b\}\}$ 4

1st 2nd ... nth

$2 \times 2 \times \cdots \times 2 = 2^n$

$A = \{a,b,c\}$

$\{\Phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$

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2. How many elements are in the power set of $A = \{a, b, c, d, e\}$?

1st 2nd 3rd 4th 5th

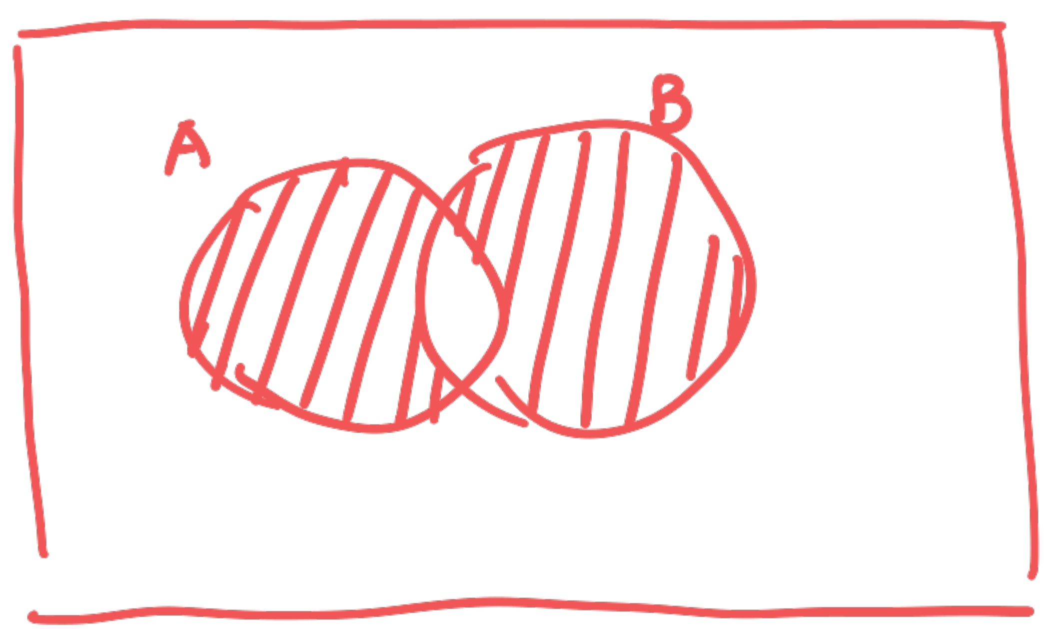
$2 \times 2 \times 2 \times 2 \times 2 = 2^5$

Exclusive OR

Exclusive OR

The **exclusive or** operator $\oplus$ for the sets A and B is definedd as

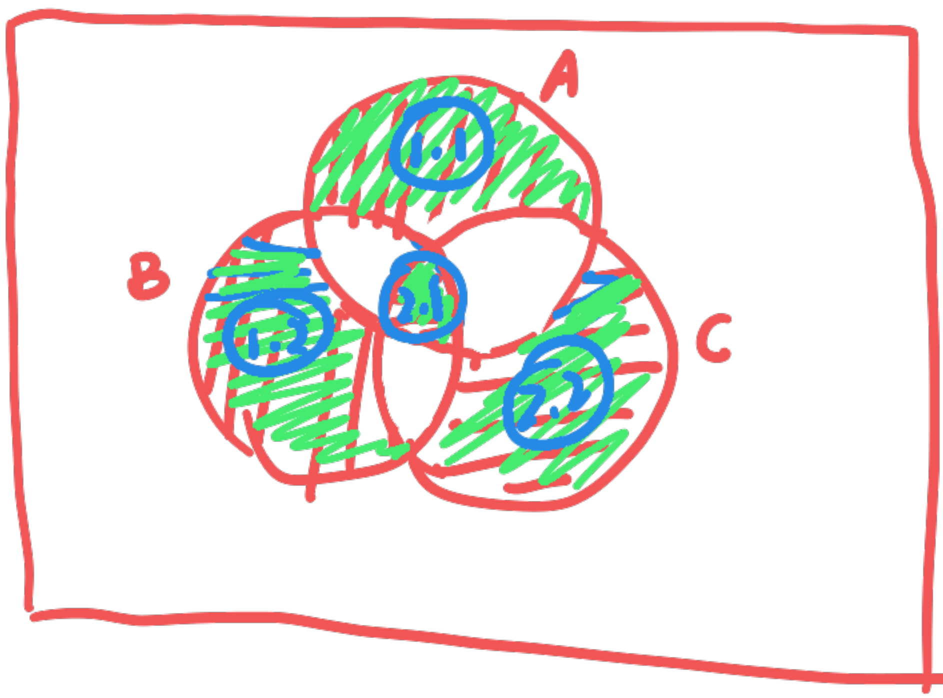
$$A \oplus B = (A \cap B^c) \cup (A^c \cap B).$$



Example

Find $A \oplus B \oplus C$ using Venn diagram.

$A \oplus B = (A \cap B^c) \cup (A^c \cap B)$



① $\{\{(A \cap B^c) \cup (A^c \cap B)\} \cap C^c\} \cup$

② $\{\{(A \cap B^c) \cup (A^c \cap B)\}^c \cap C\}$

1.1 $(A \cap B^c \cap C^c) \cup$ 1.2 $(A^c \cap B \cap C^c) \cup$

$(A \cap B^c)^c \cap (A^c \cap B)^c \cap C$

$(A^c \cup B) \cap (A \cup B^c) \cap C$

$[\{(A^c \cup B) \cap A\} \cup \{(A^c \cup B) \cap B^c\}] \cap C$

$\{(B \cap A) \cup (A^c \cap B^c)\} \cap C$

2.1 $(A \cap B \cap C) \cup$ 2.2 $(A^c \cap B^c \cap C)$

Thank You

THANK YOU!