Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 1. Introduction

1.2 Counting

GuanNan Wang
gwang01@wm.edu

What is Statistics?

- *Webster's New Collegiate Dictionary:* "A branch of mathematics dealing with the collection, analysis, interpretation, and presentation of masses of numerical data."
- *Stuart and Ord (1991):* "Statistics is the branch of the scientific method which deals with the data obtained by counting or measuring the properties of population."
- *Rice (1995):* "Statistics is essentially concerned with procedures for analyzing data, especially data that in some vague sense have a random character."
- *Freund and Walpole (1987):* "Statistics is the science of basing inferences on observed data and the entire problem of making decisions in the face of uncertainty."

All the authors imply that *"The objective of statistics is to make an inference about a population based on information contained in a sample from that population and the provide an associated measure of goodness for the inference."*

Enumeration

Enumeration

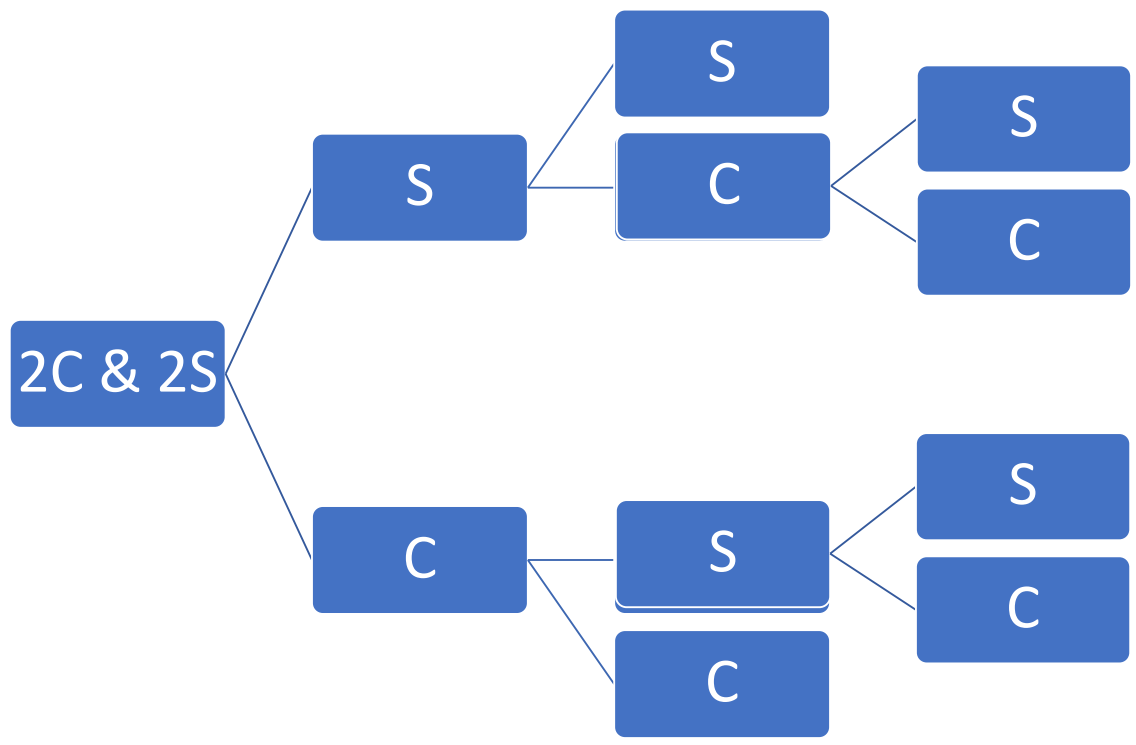
Enumeration involves listing all of the possible outcomes to an experimemt.

The Chicago Cubs and the Chicago White Sox are playing in the World Series. The best-of-seven series is tied at two games apiece. What are the possible outcomes to the series?

Enumeration

Enumeration involves listing all of the possible outcomes to an experimemt.

The Chicago Cubs and the Chicago White Sox are playing in the World Series. The best-of-seven series is tied at two games apiece. What are the possible outcomes to the series?



Multiplication Rule

Multiplication Rule

Assume that there are r decisions to be made. If there are n_1 ways to make decision 1, n_2 ways to make decision 2, . . ., n_r ways to make decision r, then there are $n_1 \times n_2 \times \cdots \times n_r$ ways to make all decisions.

How many different sequences of heads and tails are possible in 16 tosses of a fair coin?

Multiplication Rule

Assume that there are r decisions to be made. If there are n_1 ways to make decision 1, n_2 ways to make decision 2, . . ., n_r ways to make decision r, then there are $n_1 \times n_2 \times \cdots \times n_r$ ways to make all decisions.

How many different sequences of heads and tails are possible in 16 tosses of a fair coin?

Example 1

We go to Chipole to have lunch. To make an order, the first step is to choose your type of lunch, including **Burrito**, **Bowl**, **Salad**, and **Taco**. In the second step, we need to choose the protein/veggie, including **Chicken**, **Steak**, **Barbacoa**, **Carnitas**, **Sofritas**, and **Veggie**. In the third step, we can choose from **White Rice**, **Brown Rice**, and **No Rice**. Finally, we can choose from **Black Beans**, **Pinto Beans**, and **No Beans**. How many different ways we can build up our main dish at Chipole?

Example 2

1. How many ways can a family of 5 line up for photograph?

2. How many ways can a family of 5 that consists of 3 men and 2 women line up for a photograph so that men and women alternate?

Other Examples

1. How many ways can a mother give away 8 dogs to her 3 children?

2. How many ways are there to arrange the letters in the world “dynamite"?

3. How many ways can a family of 5 that consists of 3 men and 2 women line up for a photograph so that men and women alternate?

Permutations

Permutation

A **permutation** is an ordered arrangement of r objects selected from a set of n distinct oobjects without replacememt.

List the permutations from the set $\{a, b, c, d\}$ selected 2 at a time.

The number of permutations of n distinct object selected r at a time without replacement is

$$n \times (n-1) \times (n-2) \times \cdots \times (n-r+1) = \frac{n!}{(n-r)!}$$

for $r = 0, 1, 2, \ldots, n$ and n is a positive integer, and $0! = 1$.

Example 3

How many ways are there to pick a president, vice-president, and treasurer from 7 people?

A ship has 3 stands and 12 different flags to send signals. How many 3-flag signals can be sent?

What if one or two flags also constitute a signal?

Mutations of Permutation

Circular Permutations

The number of permutations of n distinct objects arranged in a circle is $(n-1)!$.

How many ways are there to seat 6 people around table for dinner?

Example 4

How many circuits can a traveling saleman make of n cities? A reverse route is not considered a unique path.

Mutations of Permutation

Nondistinct Permutations

The number of nondistinct permutatioons of n oobjects of which n_1 are of the first type, n_2 are of the second type, . . ., n_r are of the rth type, is

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

where $n_1 + n_2 + \cdots + n_r = n$

How many ways are there to line up a pair of identical twins and a set of identical triplets for a photo if identical-looking people are nondistinct?

Example 5

How many ways are there to arrange the letters in the word “door"?

How many ways are there to arrange the letters in “puppet"?

How many ways are there to arrange the letters in “wholesome"?

Combinations

Combination

A set of r objects taken from a set of n distinct objects without replacement is a **combination**.

List the combinations of 2 elements taken from $\{a, b, c, d\}$.

$(a,b) = (b,a)$
$(a,c) = (c,a)$
$(a,d) = (d,a)$
$(b,c) = (c,b)$
$(b,d) = (d,b)$
$(c,d) = (d,c)$

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = {}_nC_r = C(n,r) = C^n_r$$

The number of combinations of r objects taken without replacement from n distinct objects is

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

Example 6

How many ways are there to pick a **committee** of three people from seven “volunteers"?

$n = 7, \quad r = 3$

$$\binom{n}{r} = \binom{7}{3} = \frac{7!}{3!\,(7-3)!} = \frac{7\times 6\times 5\times 4\times 3\times 2\times 1}{3\times 2\times 1\times 4\times 3\times 2\times 1} = 35$$

♠ ♥ ♣ ♦ $\underbrace{2, 3, \cdots, 10, J, Q, K, A}_{13}$

How many ways can a five-card hand be dealt from a standard deck of playing cards?

$n = 13 \times 4 = 52$

$r = 5$

$$\binom{n}{r} = \binom{52}{5} = \frac{52!}{47!\,5!} = 2,598,960$$

Example 7

A ship has 3 stands and 12 flags to send signals. How many signals can be sent if one, two or three flags constitute a signal and the stand selected are relevant?

1-flag: $\binom{3}{1}\frac{12!}{11!} = 36$

2-flag: $+\ \binom{3}{2}\frac{12!}{10!} = 396$

3-flag: $+\ \binom{3}{3}\frac{12!}{9!} = 1320$

1752

How many ways can 14 people split into two teams of seven for a game of ultimate frisbee?

$$\frac{\binom{14}{7}}{2} = 1716$$

Teams: 1, 2, …, 7 / 8, 9, …, 14 = 8, 9, …, 14 / 1, 2, …, 7

Properties of Combination

1. The well-known **binomial theorem** can be used to expand quantities such as
$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r,$$
where $\binom{n}{r}$ is often referred to as a “binomial coefficient".
2. Several results associated with the binomial coefficients:
   2.1 Symmetry: $\binom{n}{r} = \binom{n}{n-r}$, for $r = 0, 1, \ldots, n$, and n is a positive integer.
   2.2 $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$
   2.3 $\sum_{r=0}^{k} \binom{m}{r} \binom{n}{k-r} = \binom{m+n}{k}$
3. The binomial coefficient $\binom{n}{r}$ is defined to be 0 when $r < 0$ or $r > n$.

$n=1$ $$(x+y)^1 = x + y = \binom{1}{1} x^1 y^0 + \binom{1}{0} x^0 y^1$$

$n=2$ $$(x+y)^2 = x^2 + 2xy + y^2 = \binom{2}{2} x^2 + \binom{2}{1} x^1 y^1 + \binom{2}{0} x^0 y^2$$

$n=3$ $$(x+y)^3 = x^3 + 3x^2y + 3xy^2 + y^3 = \binom{3}{3} x^3 y^0 + \binom{3}{2} x^2 y^1 + \binom{3}{1} x^1 y^2 +$$
$$\binom{3}{0} x^0 y^3$$

$n=4$ $$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$$
$$= \binom{4}{4} x^4 y^0 + \binom{4}{3} x^3 y^1 + \binom{4}{2} x^2 y^2 + \binom{4}{1} x^1 y^3 + \binom{4}{0} y^4$$

$\vdots$

$n=n$ $$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r$$

binomial coefficient.

$$\binom{n}{r} = \binom{n}{n-r}$$

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

$$\binom{n}{n-r} = \frac{n!}{(n-r)!\,\{n-(n-r)\}!} = \frac{n!}{(n-r)!\,r!}$$

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$$

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

$$\binom{n-1}{r} + \binom{n-1}{r-1} = \frac{(n-1)!}{r!\,[n-1-r)!} + \frac{(n-1)!}{(r-1)!\,\{(n-1)-(r-1)\}!}$$

$$= \frac{(n-1)!}{r!\,(n-r-1)!} + \frac{(n-1)!}{(r-1)!\,(n-r)!}$$

$$= \frac{(n-r)(n-1)! + r(n-1)!}{r!\,(n-r)!}$$

$$= \frac{n(n-1)!}{r!\,(n-r)!} = \frac{n!}{r!\,(n-r)!}$$

$$\sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r} = \binom{m+n}{k}$$

$$(1+x)^{m+n} = (1+x)^m (1+x)^n$$

examine the coefficient of x^k

$$(1+x)^{m+n} = \sum_{k=0}^{m+n} \binom{m+n}{k} x^k 1^{(m+n-k)} = \sum_{k=0}^{m+n} \binom{m+n}{k} x^k$$

$$(1+x)^m (1+x)^n = \sum_{i=0}^{k} \binom{m}{i} x^i \binom{n}{k-i} x^{k-i}$$

i out of k (k-i) out of k

$$= \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i} x^k$$

$$\binom{m+n}{k} = \sum_{i=0}^{k} \binom{m}{i}\binom{n}{k-i}$$

Multinomial

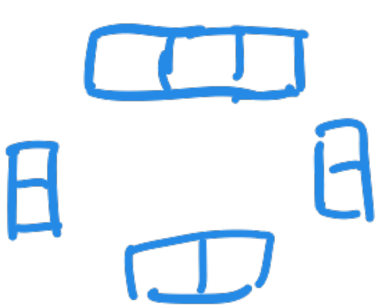
Multinomial

The number of ways of partitioning a set of n distinct objects into k subsets with n_1 in the first subset, n_2 in the second subset, ..., n_k in the kth subset is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!n_2!\ldots n_k!}$$

where $n_1 + n_2 + \cdots + n_k = n$.

The Glen family consists of 9 people. How many arrangements are there for them to watch the nightly news seated on four sofas: one that seats three and the others seat two?



9

$$\binom{9}{3, 2, 2, 2} = \frac{9!}{3!\,2!\,2!\,2!} = 7560$$

Example 8

How many ways are there to select 4 billiard balls from a bag containing the 15 balls numbered $1, 2, \ldots, 15$ under the following scenarios?

	Without replacement	With replacement
Ordered sample	① 32760	③ 50625
Unordered sample	② 1365	

① $n=15$ $r=4$. without rep. / order

$$_nP_r = \frac{15!}{(15-4)!} = 15\times14\times13\times12 = 32760$$

② $n=15$ $r=4$ without rep / unorder

$$\binom{15}{4} = \frac{15!}{4!\,(15-4)!} = \frac{15!}{4!\,11!} = 1365$$

③ $n=15$. $r=4$ with rep ordered.

1st × 2nd × 3rd × 4th

$$15 \times 15 \times 15 \times 15 = 50625$$

④ $n=15$. $r=4$. with rep unordered.

4.1 no rep. $\binom{15}{4} = 1365$ +

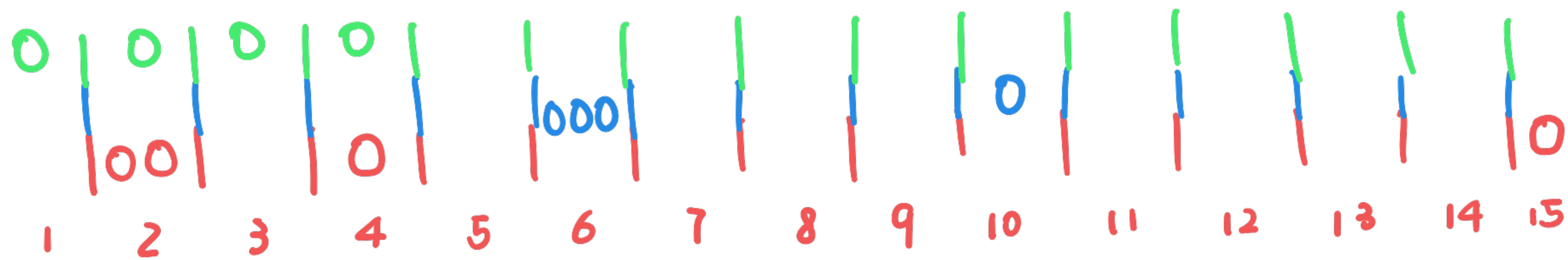
4.2 1 ball rep twice. 2 diff. $\binom{15}{1}\binom{14}{2} = 1365$ +

4.3 1 ball rep three time 1 diff $\binom{15}{1}\binom{14}{1} = 210$ +

4.4 2 ball rep twice. $\binom{15}{2} = 105$ +

4.5 1 ball rep four times $\binom{15}{1} = 15$

} 3060



① 2 2 4 15 ② 6 6 6 10 ③ 1 2 3 4

$\binom{14+4}{4} = \binom{18}{4} = 3060$

THANK YOU!