Department of Mathematics
College of William & Mary

MATH 451/551

Chapter 1. Introduction

1.2 Counting

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What is Statistics?

analysis { ① descriptive statistic ② inferential statistic

- *Webster's New Collegiate Dictionary:* "A branch of mathematics dealing with the ① collection, ② analysis, ③ interpretation, and presentation of masses of numerical data."

  sample —inference→ population

- *Stuart and Ord (1991):* "Statistics is the branch of the scientific method which deals with the data obtained by counting or measuring the properties of population."

  population: entire group of subject
  sample: subset of the population

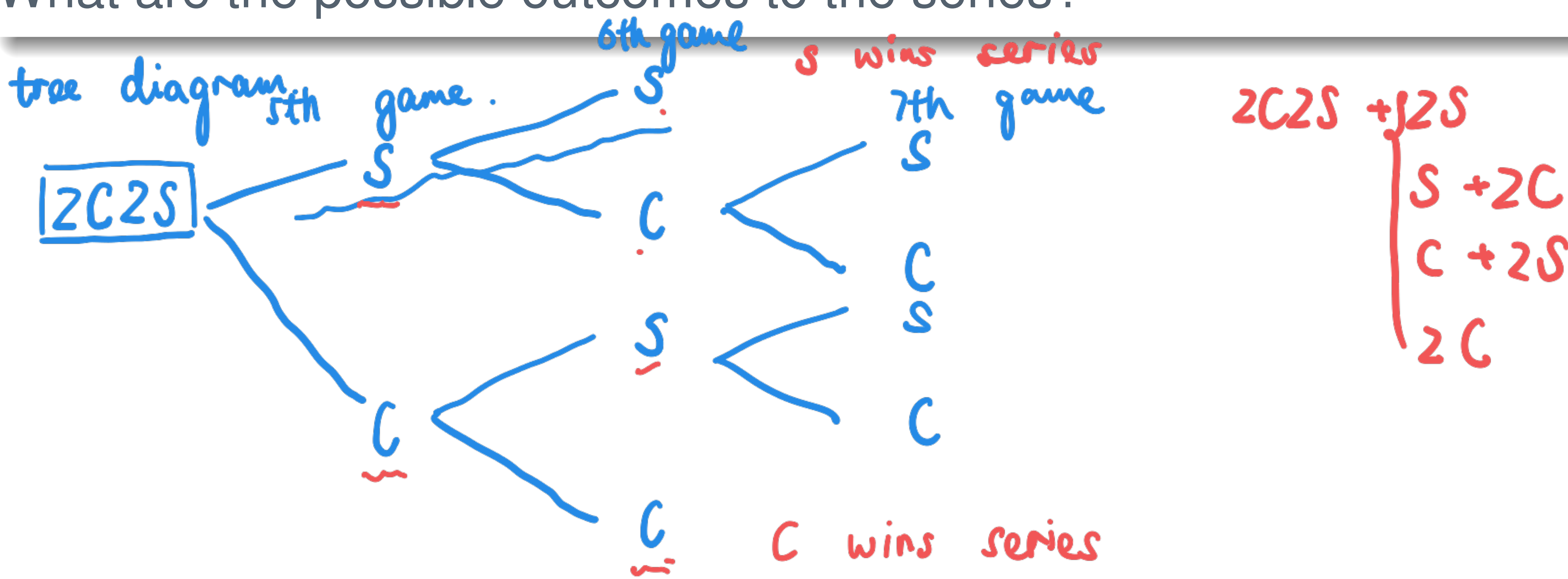
- *Rice (1995):* "Statistics is essentially concerned with procedures for analyzing data, especially data that in some vague sense have a random character."
- *Freund and Walpole (1987):* "Statistics is the science of basing inferences on observed data and the entire problem of making decisions in the face of uncertainty."

All the authors imply that *"The objective of statistics is to make an inference about a population based on information contained in a sample from that population and the provide an associated measure of goodness for the inference."*

Enumeration

Enumeration involves listing all of the possible outcomes to an experimemt.

The Chicago Cubs and the Chicago White Sox are playing in the World Series. The best-of-seven series is tied at two games apiece. What are the possible outcomes to the series?



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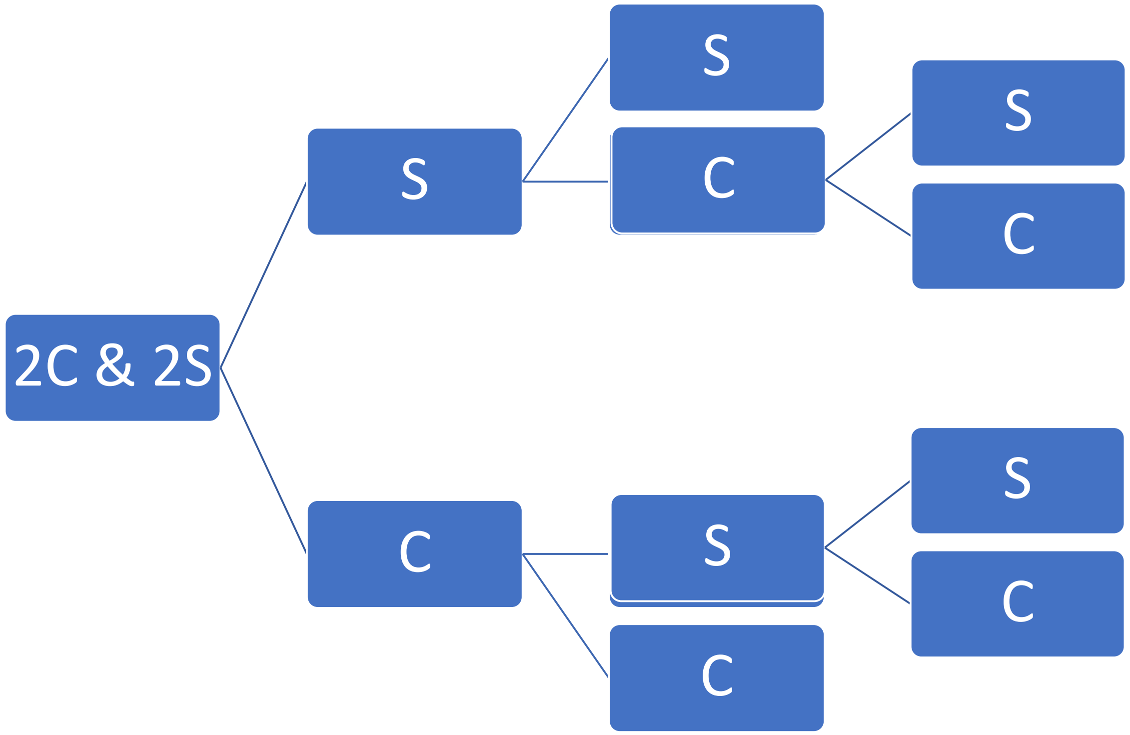

① multiplication rule.
② Permutation
③ Combination

Multiplication Rule

Assume that there are r decisions to be made. If there are n_1 ways to make decision 1, n_2 ways to make decision 2, . . ., n_r ways to make decision r, then there are $n_1 \times n_2 \times \cdots \times n_r$ ways to make all decisions.

How many different sequences of heads and tails are possible in 16 tosses of a fair coin?

1st (H, T) 2nd (H, T) . . . . 16th (H, T)

$2 \times 2 \times \cdots \times 2 = 2^{16}$

Multiplication Rule

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How many different sequences of heads and tails are possible in 16 tosses of a fair coin?

Example 1

We go to Chipole to have lunch. To make an order, the first step is to choose your type of lunch, including ①**Burrito**, ②**Bowl**, ③**Salad**, and ④**Taco**. In the second step, we need to choose the protein/veggie, including ①**Chicken**, ②**Steak**, ③**Barbacoa**, ④**Carnitas**, ⑤**Sofritas**, and ⑥**Veggie**. In the third step, we can choose from ①**White Rice**, ②**Brown Rice**, and ③**No Rice**. Finally, we can choose from ①**Black Beans**, ②**Pinto Beans**, and ③**No Beans**. How many different ways we can build up our main dish at Chipole?

1st 2nd 3rd 4th

$4 \times 6 \times 3 \times 3 = 216$

Example 2

1. How many ways can a family of 5 line up for photograph?

   1st 2nd 3rd 4th 5th

   $5 \times 4 \times 3 \times 2 \times 1 = 120$

2. How many ways can a family of 5 that consists of 3 men and 2 women line up for a photograph so that men and women alternate?

   1st 2nd 3rd 4th 5th

   M W M W M

   $3 \times 2 \times 2 \times 1 \times 1 = 12$

Other Examples

1. How many ways can a mother give away 8 dogs to her 3 children?

2. How many ways are there to arrange the letters in the world “dynamite"?

3. How many ways can a family of 5 that consists of 3 men and 2 women line up for a photograph so that men and women alternate?

with replacement $n \times n \times \cdots \times n = n^r$
without replacement $n \times (n-1) \times \cdots \times (n-r+1)$
r decision.

Permutation

A **permutation** is an ordered arrangement of r objects selected from a set of n distinct oobjects without replacememt.

List the permutations from the set $\{a, b, c, d\}$ selected 2 at a time.

(a,b) (b,a) (c,a) (d,a)
(a,c) (b,c) (c,b) (d,b)
(a,d) (b,d) (c,d) (d,c)

12.

1st 2nd
$4 \times 3 = 12$

$_nP_r = P(n,r) = P^n_r$

The number of permutations of n distinct object selected r at a time without replacement is

$$\underset{\text{1st}}{n} \times \underset{\text{2nd}}{(n-1)} \times (n-2) \times \cdots \times \underset{r\text{-th}}{(n-r+1)} = \frac{n!}{(n-r)!}$$

for $r = 0, 1, 2, \ldots, n$ and n is a positive integer, and $0! = 1$.

Example 3

How many ways are there to pick a president, vice-president, and treasurer from 7 people?

$n=7.\ r=3.$

$${}_7P_3 = \frac{n!}{(n-r)!} = \frac{7!}{(7-4)!} =$$

President × Vice-president × Treasurer.

$$7 \times 6 \times 5 = 210$$

A ship has 3 stands and 12 different flags to send signals. How many 3-flag signals can be sent?

$${}_{12}P_3 = \frac{12!}{(12-3)!} = 12 \times 11 \times 10 = 1320$$

What if one or two flags also constitute a signal?

1-flag. 2-flag 3-flag

$${}_{12}P_1 + {}_{12}P_2 + {}_{12}P_3 = 12 + 12 \times 11 + 12 \times 11 \times 10 = 1464$$

Mutations of Permutation

circular
non-distinct

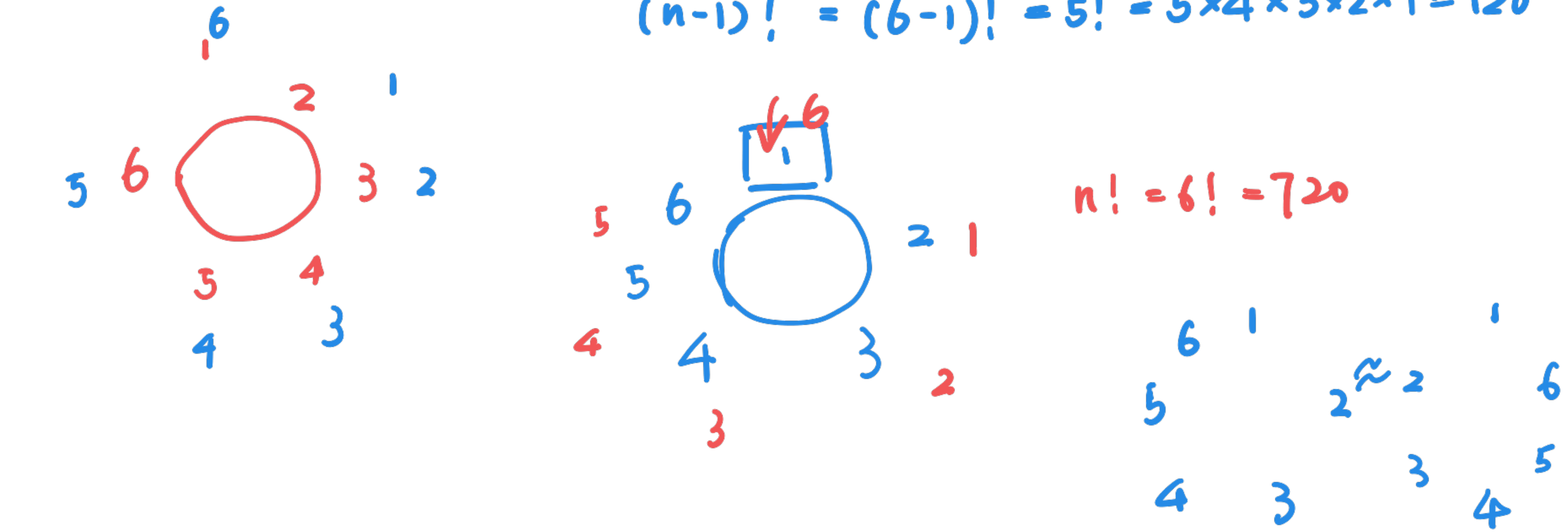
Circular Permutations

The number of permutations of n distinct objects arranged in a circle is $(n-1)!$.

How many ways are there to seat 6 people around table for dinner?

1. 2. 3. 4. 5. 6

$(n-1)! = (6-1)! = 5! = 5\times4\times3\times2\times1 = 120$



$n! = 6! = 720$

Example 4

How many circuits can a traveling saleman make of *n* cities? A reverse route is not considered a unique path.

Mutations of Permutation

Nondistinct Permutations

The number of nondistinct permutations of n objects of which n_1 are of the first type, n_2 are of the second type, ..., n_r are of the rth type, is

$$\frac{n!}{n_1!n_2!\cdots n_r!}$$

where $n_1 + n_2 + \cdots + n_r = n$

How many ways are there to line up a pair of identical twins and a set of identical triplets for a photo if identical-looking people are nondistinct?

$2 + 3 = n = 5$

tw 1 2 tri 1 2 3

$$\frac{n!}{2!\,3!} = \frac{5!}{2!\,3!} = \frac{5\times4\times3\times2\times1}{2\times1\times3\times2\times1} = 10$$

Example 5

How many ways are there to arrange the letters in the word “door"?

1 d
2 o
1 r
4 letter

$$\frac{4!}{1!\times 2!\times 1!} = 12$$

How many ways are there to arrange the letters in “puppet"?

3 p
1 u
1 e
1 t
6

$$\frac{6!}{3!\times 1!\times 1!\times 1!} = \frac{720}{6} = 120$$

How many ways are there to arrange the letters in “wholesome"?

Combinations

Combination

A set of r objects taken from a set of n distinct objects without replacement is a **combination**.

List the combinations of 2 elements taken from $\{a, b, c, d\}$.

The number of combinations of r objects taken without replacement from n distinct objects is

$$\binom{n}{r} = \frac{n!}{(n-r)!r!}.$$

Example 6

How many ways are there to pick a **committee** of three people from seven “volunteers"?

How many ways can a five-card hand be dealt from a standard deck of playing cards?

Example 7

A ship has 3 stands and 12 flags to send signals. How many signals can be sent if one, two or three flags constitute a signal and the stand selected are relevant?

How many ways can 14 people split into two teams of seven for a game of ultimate frisbee?

Properties of Combination

1. The well-known **binomial theorem** can be used to expand quantities such as
$$(x+y)^n = \sum_{r=0}^{n} \binom{n}{r} x^{n-r} y^r,$$
where $\binom{n}{r}$ is often referred to as a “binomial coefficient".
2. Several results associated with the binomial coefficients:
   2.1 Symmetry: $\binom{n}{r} = \binom{n}{n-r}$, for $r = 0, 1, \ldots, n$, and n is a positive integer.
   2.2 $\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}$
   2.3 $\sum_{r=0}^{k} \binom{m}{r}\binom{n}{k-r} = \binom{m+n}{k}$
3. The binomial coefficient $\binom{n}{r}$ is defined to be 0 when $r < 0$ or $r > n$.

Multinomial

Multinomial

The number of ways of partitioning a set of n distinct objects into k subsets with n_1 in the first subset, n_2 in the second subset, ..., n_k in the kth subset is

$$\binom{n}{n_1, n_2, \ldots, n_k} = \frac{n!}{n_1!n_2!\ldots n_k!}$$

where $n_1 + n_2 + \cdots + n_k = n$.

The Glen family consists of 9 people. How many arrangements are there for them to watch the nightly news seated on four sofas: one that seats three and the other seat two?

Example 8

How many ways are there to select 4 billiard balls from a bag containing the 15 balls numbered $1, 2, \ldots, 15$ under the following scenarios?

	Without replacement	With replacement
Ordered sample		
Unordered sample		

THANK YOU!