Uniqueness of the Solutions of Some Completion Problems

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Abstract

We determine the conditions for uniqueness of the solutions of several completion problems including the positive semi-definite completion, the distance matrix completion, and the contractive matrix completion. The conditions depend on the existence of a positive semi-definite matrix satisfying certain linear constraints. Numerically, such conditions can be checked by existing computer software such as semi-definite programming routines. Some numerical examples are given to illustrate the results, and show that some results in a recent paper of Alfakih are incorrect. Discrepancies in the proof of Alfakih are explained.

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1 Introduction

In the study of completion problems, one considers a partially specified matrix and tries to fill in the missing entries so that the resulting matrix has some specific properties such as being invertible, having a specific rank, being positive semi-definite, etc. One can ask the following general problems:

(a) Determine whether a completion with the desired property exists.

(b) Determine all completions with the desired property.

(c) Determine whether there is a unique completion with the desired property.

(d) Determine the "best" completion with a desired property under certain criteria.

See [5] for general background of completion problems.

In [1], the author raised the problem of determining the condition on an $n \times n$ partial matrix A under which there is a unique way to complete it to a Euclidean distance squared

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(EDS) matrix, i.e., a matrix of the form $(\|x_i - x_j\|^2)_{1\le i,j\le n}$ for some $x_1, \ldots, x_n \in \mathbf{R}^k$. In this paper, we give a complete answer to this problem. It turns out that the desired uniqueness condition can be determined by the existence of a positive semi-definite matrix satisfying certain linear constraints. Such a condition can be checked by existing computer software such as the semi-definite programming routines; see [7, 8].

Our paper is organized as follows. In Section 2, we first obtain a necessary and sufficient condition for an $n\times n$ partial matrix A to have a unique positive semi-definite completion. We then use the result to deduce the conditions for the uniqueness of the EDS matrix completion and the contractive matrix completion problem. (Recall that a matrix is contractive if its operator norm is at most one.) Furthermore, we describe an algorithm to check the conditions in our results, and how to use existing software to check the conditions numerically. In Section 3, we illustrate our results by several numerical examples. In Section 4, we show that some results in [1] are not accurate. Discrepancies in the proofs in [1] are explained.

In our discussion, we denote by $\mathbf{S}_n$ the space of $n \times n$ symmetric matrices, EDS_n the set of $n \times n$ EDS matrices, and PD_n (respectively, PSD_n) the set of positive (semi-)definite matrices in $\mathbf{S}_n$. The standard inner product on $\mathbf{S}_n$ is defined by $(X, Y) = \mathrm{tr}\,(XY)$. The standard basis of $\mathbf{R}^n$ will be denoted by $\{e_1, \ldots, e_n\}$ and we let $e = e_1 + \cdots + e_n$.

2 Uniqueness of completion problems

We consider problems in the following general settings.

Let $\mathcal{M}$ be a matrix space, and $\mathcal{S}$ a subspace of $\mathcal{M}$. Suppose $\mathcal{P}$ is a subset of $\mathcal{M}$ with certain desirable properties. Given $A \in \mathcal{M}$, we would like to determine $X \in \mathcal{S}$ so that

$$A + X \in \mathcal{P}.$$

In particular, we are interested in the condition for the uniqueness of $X \in \mathcal{S}$ such that $A + X \in \mathcal{P}$. We will always assume that there is an $X_0 \in \mathcal{S}$ such that $A + X_0 \in \mathcal{P}$, and study the condition under which X_0 is the only matrix in $\mathcal{S}$ satisfying $A + X_0 \in \mathcal{P}$. We can always assume that $X_0 = 0$ by replacing A by $A + X_0$.

To recover the completion problem, suppose a partial matrix is given. Let A be an arbitrary completion of the partial matrix, say, set all unspecified entries to 0. Let $\mathcal{S}$ be the space of matrices with zero entries at the specified entries of the given partial matrix. Suppose $\mathcal{P}$ is a subset of $\mathcal{M}$ with the desired property such as being invertible, having a specific rank, being positive semi-definite, etc. Then completing the partial matrix to a matrix in $\mathcal{P}$ is the same as finding $X \in \mathcal{S}$ such that $A + X \in \mathcal{P}$.

We begin with the following result concerning the uniqueness of the positive semi-definite completion problem.

Proposition 2.1 *Let $A \in \mathrm{PSD}_n$, and $\mathcal{S}$ be a subspace of $\mathbf{S}_n$. Suppose V is orthogonal such that $V^tAV = \mathrm{diag}\,(d_1, \ldots, d_r, 0, \ldots, 0)$, where $d_1 \ge \cdots \ge d_r > 0$. If $P \in \mathcal{S}$ satisfies $A + P \in \mathrm{PSD}_n$, then*

$$X = V^tPV = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \tag{1}$$

with

$$X_{22} \in \mathrm{PSD}_{n-r} \quad \textit{and} \quad \mathrm{rank}\,(X_{22}) = \mathrm{rank}\,([X_{21}\ X_{22}]). \tag{2}$$

Conversely, if there is a $P \in \mathcal{S}$ *such that* (1) *and* (2) *hold then there is* $\varepsilon > 0$ *such that* $A + \delta P \in \mathrm{PSD}_n$ *for all* $\delta \in [0, \varepsilon]$.

Remark 2.2 Note that in Proposition 2.1 one needs only find an orthogonal matrix V such that $V^tAV = D \oplus 0$ for a positive definite matrix D, i.e., the last $n-r$ columns of V form an orthonormal basis for the kernel of A. The statement and the proof of the result will still be valid.

Proof of Proposition 2.1. Suppose $A + P \in \mathrm{PSD}_n$. Let $X = V^tPV$ be partitioned as in (1). We have $X_{22} \in \mathrm{PSD}_{n-r}$ because

$$\begin{bmatrix} D + X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \in \mathrm{PSD}_n \qquad \text{with } D = \mathrm{diag}\,(d_1, \ldots, d_r).$$

Let W be orthogonal such that $W^tX_{22}W = \mathrm{diag}\,(c_1, \ldots, c_s, 0, \ldots, 0)$ with $c_1 \ge \cdots \ge c_s > 0$. If $\widetilde{W} = I_r \oplus W$, then

$$\widetilde{W}^tV^t(A + P)V\widetilde{W} = \begin{bmatrix} D + X_{11} & Y_{12} \\ Y_{21} & W^tX_{22}W \end{bmatrix}.$$

Since $A + P \in \mathrm{PSD}_n$, we see that only the first s rows of Y_{21} can be nonzero. Thus,

$$\mathrm{rank}\,([X_{21}\ X_{22}]) = \mathrm{rank}\,([Y_{21}\ W^tX_{22}W]) = s = \mathrm{rank}\,(X_{22}).$$

Conversely, suppose there is a $P \in \mathcal{S}$ such that (1) and (2) hold. Then for sufficiently large $\eta > 0$, $\eta D + X_{11}$ is positive definite. Moreover, if

$$T = \begin{bmatrix} I_r & -(\eta D + X_{11})^{-1}X_{12} \\ 0 & I_{n-r} \end{bmatrix},$$

then

$$T^tV^t(\eta A + X)VT = (\eta D + X_{11}) \oplus [X_{22} - X_{21}(\eta D + X_{11})^{-1}X_{12}].$$

Since $\mathrm{rank}\,([X_{21}\, X_{22}]) = \mathrm{rank}\,(X_{22})$, for sufficiently large $\eta > 0$ we have

$$X_{22} - X_{21}(\eta D/2)^{-1}X_{12} \in \mathrm{PSD}_{n-r} \quad \text{and} \quad (\eta D/2)^{-1} - (\eta D + X_{11})^{-1} \in \mathrm{PD}_r.$$

Hence, under the positive semi-definite ordering $\succeq$, we have

$$X_{22} - X_{21}(\eta D + X_{11})^{-1}X_{12} \succeq X_{22} - X_{21}(\eta D/2)^{-1}X_{12} \succeq 0_{n-r}.$$

Thus, for sufficiently large η, we have $A + P/\eta \in \mathrm{PSD}_n$. □

By Proposition 2.1, the zero matrix is the only element P in $\mathcal{S}$ such that $A + P \in \mathrm{PSD}_n$ if and only if the zero matrix is the only element X in $\mathcal{S}$ such that $V^tXV = (X_{ij})_{1 \le i,j \le 2}$ with $X_{22} \in \mathrm{PSD}_{n-r}$ and $\mathrm{rank}\,(X_{22}) = \mathrm{rank}\,([X_{21}\ X_{22}])$. This condition can be checked by the following algorithm.

Algorithm 2.3 *Let $\mathcal{S}$ be a subspace of* $\mathbf{S}_n$*, and* $A \in \mathrm{PSD}_n$.

Step 1 Construct a basis $\{X_1, \ldots, X_k\}$ for $\mathcal{S}$.

Step 2 Determine the dimension l of the space

$$\tilde{\mathcal{S}} = \left\{ [\, X_{21} \quad X_{22} \,] : V^t X V = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \text{ with } X \in \mathcal{S} \right\}.$$

If $k > l$, then there is a nonzero $P \in \mathcal{S}$ such that $V^t P V = P_1 \oplus 0_{n-r}$ and $A + P \in \mathrm{PSD}_n$ and so the completion is not unique. Otherwise, go to Step 3.

Step 3 Determine whether there are real numbers $a_0, a_1, \ldots, a_k$ such that

$$Q = a_0 A + a_1 X_1 + \cdots + a_k X_k \in \mathrm{PSD}_n$$

with $(0_r \oplus I_{n-r}, V^t Q V) = 1$.

If such a matrix Q exists, then there is a nonzero $P \in \mathcal{S}$ such that $A + P \in \mathrm{PSD}_n$. Otherwise, we can conclude that 0_n is the only matrix P in $\mathcal{S}$ such that $A + P \in \mathrm{PSD}_n$.

(Note that numerically Step 3 can be checked by existing software such as semi-definite programming routines.)

Explanation of the algorithm

Note that in Step 2, the condition $k > l$ holds if and only if there is a nonzero matrix $P \in \mathcal{S}$ such that $V^t P V = P_1 \oplus 0_{n-r}$ and $A + P \in \mathrm{PSD}_n$. To see this, let $V = [V_1|V_2]$ such that V_1 is $n \times r$. Then

$$\tilde{\mathcal{S}} = \{V_2^t X V : X \in \mathcal{S}\}$$

and $\{V_2^t X_1 V, \ldots, V_2^t X_k V\}$ is a spanning set of $\tilde{\mathcal{S}}$.

If $k > l$, then there is a nonzero real vector $(a_1, \ldots, a_k)$ such that $a_1 V_2^t X_1 V + \cdots + a_k V_2^t X_k V = 0_{n-r,n}$. Since $X_1, \ldots, X_k$ are linearly independent, $P = a_1 X_1 + \cdots + a_k X_k$ is nonzero. Clearly, $X = V^t P V$ has the form $X_{11} \oplus O_{n-r}$. By Proposition 2.1, there is $\delta > 0$ such that $A + \delta P \in \mathrm{PSD}_n$.

Conversely, if there is a nonzero matrix $P \in \mathcal{S}$ such that $V^t P V = P_1 \oplus 0_{n-r}$ and $A + P \in \mathrm{PSD}_n$, then there is a nonzero real vector $(a_1, \ldots, a_k)$ such that $P = a_1 X_1 + \cdots + a_k X_k$ so that $a_1 V_2^t X_1 V + \cdots + a_k V_2^t X_k V = 0_{n-r,n}$. Hence, $\tilde{\mathcal{S}}$ has dimension less than k.

So, if $k = l$, and if there is a nonzero $P \in \mathcal{S}$ such that $A + P \in \mathrm{PSD}_n$, then $V_2^t P V$ cannot be zero. By Proposition 2.1, $V_2^t P V_2$ is nonzero, and Step 3 will detect such a matrix P if it exists.

By Proposition 2.1 and its proof, we have the following corollary.

Corollary 2.4 *Suppose $\mathcal{S} \subseteq \mathbf{S}_n$, $A \in \mathrm{PSD}_n$, and the orthogonal matrix V satisfy the hypotheses of Proposition* 2.1.

(a) *If $A \in \mathrm{PD}_n$, then for any $P \in \mathcal{S}$ and sufficiently small $\delta > 0$, we have $A + \delta P \in \mathrm{PD}_n$.*

(b) *If there is a $P \in \mathcal{S}$ such that the matrix X_{22} in* (1) *is positive definite, then $A + \delta P \in \mathrm{PD}_n$ for sufficiently small $\delta > 0$.*

Remark 2.5 To use condition (b) in Corollary 2.4, one can focus on the matrix space

$$\mathcal{T} = \{V_2^t P V_2 : P \in \mathcal{S}\} \subseteq \mathbf{S}_{n-r},$$

where V_2 is obtained from V by removing its first r columns. Note that PD_m is the interior of PSD_m, and PSD_m is a self-dual cone, i.e.,

$$\mathrm{PSD}_m = \{Y \in \mathbf{S}_m : (Y, Z) \geq 0 \text{ for all } Z \in \mathrm{PD}_m\}.$$

By the theorem of alternative (e.g., see [6]), $\mathcal{T} \cap \mathrm{PD}_{n-r} \neq \emptyset$ if and only if

$$\mathcal{T}^{\perp} \cap \mathrm{PSD}_{n-r} = 0. \tag{3}$$

One can use standard semi-definite programming routines to check condition (3).

Here is another consequence of Proposition 2.1.

Corollary 2.6 *Suppose $\mathcal{S} \subseteq \mathbf{S}_n$, $A \in \mathrm{PSD}_n$, $\mathrm{rank}\,(A) = n - 1$ and the orthogonal matrix V satisfy the hypotheses of Proposition* 2.1 *If $\mathcal{S}$ has dimension larger than $n - 1$, then there is a $P \in \mathcal{S}$ such that $A + \delta P \in \mathrm{PSD}_n$ for all sufficiently small $\delta > 0$.*

Proof. If there is a $P \in \mathcal{S}$ such that VPV^t has nonzero (n, n) entry, we may assume that it is positive; otherwise replace P by $-P$. Then by Proposition 2.1 $A + \delta P \in \mathrm{PSD}_n$ for sufficiently small $\delta > 0$. Suppose VPV^t always has zero entry at the (n, n) position. Since $\mathcal{S}$ has dimension at least n, there exists a nonzero $P \in \mathcal{S}$ such that the last column of VPV^t are zero. So, $A + \delta P \in \mathrm{PSD}_n$ for sufficiently small $\delta > 0$. □

Next, we use Proposition 2.1 to answer the question raised in [1].

Proposition 2.7 *Let $\mathbf{S}_n^0$ be the subspace of matrices in $\mathbf{S}_n$ with all diagonal entries equal to zero. Let $A \in \mathrm{EDS}_n$, and $\mathcal{S}$ be a subspace of $\mathbf{S}_n^0$. Then there is an $n \times (n - 1)$ matrix U such that $U^t e = 0$, $U^t U = I_{n-1}$, and $-U^t A U = \mathrm{diag}\,(d_1, \ldots, d_r, 0, \ldots, 0)$, where $d_1 \geq \cdots \geq d_r > 0$. Moreover, there is a nonzero matrix $P \in \mathcal{S}$ such that $A + P \in \mathrm{EDS}_n$ if and only if there is nonzero matrix $P \in \mathcal{S}$ such that*

$$X = U^t P U = \begin{bmatrix} X_{11} & X_{12} \\ X_{21} & X_{22} \end{bmatrix} \tag{4}$$

with $X_{22} \in \mathrm{PSD}_{n-1-r}$ and $\mathrm{rank}\,(X_{22}) = \mathrm{rank}\,([X_{21}\ X_{22}])$.

Proof. By the result in [4], for any $n \times (n - 1)$ matrix W such that $W^t W = I_{n-1}$, the mapping $X \mapsto -\frac{1}{2} W^t X W$ is a linear isomorphism from $\mathbf{S}_n^0$ to $\mathbf{S}_{n-1}$ such that the cone EDS_n is mapped onto PSD_{n-1}. Since $-\frac{1}{2} W^t A W$ is positive semi-definite, there is an $(n-1) \times (n-1)$

orthogonal matrix V such that $-\frac{1}{2}V^tW^tAWV = \mathrm{diag}\,(d_1,\ldots,d_r,0,\ldots,0)$, where $d_1 \geq \cdots \geq d_r > 0$. Evidently, $U = WV$ satisfies the asserted condition.

Now, the existence of a nonzero $P \in \mathcal{S}$ such that $A+P \in \mathrm{EDS}_n$ is equivalent to the existence of a nonzero $X \in \{-\frac{1}{2}U^tPU : P \in \mathcal{S}\}$ such that $-\frac{1}{2}U^tAU + X \in PSD_{n-1}$. One can apply Proposition 2.1 to get the conclusion. □

Using Corollaries 2.4 and 2.6, we have the following corollary concerning unique EDS matrix completion. Part (a) in the following was also observed in [1, Theorem 3.1].

Corollary 2.8 *Use the notations in Proposition* 2.7*.*

(a) *If U^tAU has rank $n-1$, then for any $P \in \mathcal{S}$ and sufficiently small $\delta > 0$, we have $A + \delta P \in \mathrm{EDS}_n$*

(b) *If there is a $P \in \mathcal{S}$ such that the matrix X_{22} in* (4) *is positive definite, then $A + \delta P \in \mathrm{EDS}_n$ for sufficiently small $\delta > 0$.*

(c) *If $\mathrm{rank}\,(U^tAU) = n-2$ and $\mathcal{S}$ has dimension larger than $n-2$, then there is a $P \in \mathcal{S}$ such that $A + \delta P \in \mathrm{EDS}_n$ for all sufficiently small $\delta > 0$.*

Note that Proposition 2.1 is also valid for the real space $\mathbf{H}_n$ of $n \times n$ complex Hermitian matrices. Moreover, our techniques can be applied to other completion problems on the space $\mathbf{M}_{m,n}$ of $m \times n$ complex matrices that can be formulated in terms of positive semi-definite matrices. For instance, for any $B \in \mathbf{M}_{m,n}$, the operator norm $\|B\| \leq 1$ if and only if

$$\begin{bmatrix} I_m & B \\ B^* & I_n \end{bmatrix} \in \mathrm{PSD}_{m+n}.$$

As a result, if $\tilde{\mathcal{S}}$ is a subspace of $\mathbf{M}_{m,n}$, and $\tilde{A} \in \mathbf{M}_{m,n}$ such that $\|\tilde{A}\| \leq 1$, we can let

$$A = \begin{bmatrix} I_m & \tilde{A} \\ \tilde{A}^* & I_n \end{bmatrix} \in \mathrm{PSD}_{m+n},$$

and $\mathcal{S}$ be the subspace of $\mathbf{H}_{m+n}$ consisting of matrices of the form

$$P = \begin{bmatrix} 0_m & \tilde{P} \\ \tilde{P}^* & 0_n \end{bmatrix}$$

with $\tilde{P} \in \tilde{\mathcal{S}}$. Then there is a $\tilde{P} \in \tilde{\mathcal{S}}$ such that $\|\tilde{A} + \tilde{P}\| \leq 1$ if and only if there is a $P \in \mathcal{S}$ such that $A + P \in \mathrm{PSD}_{m+n}$. We can then apply Proposition 2.1 to determine the uniqueness condition.

3 Numerical examples

We illustrate how to use our results and algorithm in the previous section in the following. We begin with the positive semi-definite matrix completion problem in the general setting.

Example 3.1 Let

$$A_1 = I_6 \oplus [0], \quad A_2 = I_5 \oplus 0_2, \quad A_3 = I_4 \oplus 0_3 \quad \text{and} \quad A_4 = I_3 \oplus 0_4.$$

Let $b = 1/\sqrt{2}$ and $\mathcal{S} = \text{span}\,\{X_1, X_2, X_3, X_4\}$ where

$$X_1 = \begin{bmatrix} -1 & 0 & -1 & 0 & -1 & -b & b \\ 0 & 1 & 0 & 1 & 0 & b & b \\ -1 & 0 & -1 & 0 & -1 & -b & b \\ 0 & 1 & 0 & 1 & 0 & b & b \\ -1 & 0 & -1 & 0 & -1 & -b & b \\ -b & b & -b & b & -b & 0 & 1 \\ b & b & b & b & b & 1 & 0 \end{bmatrix}, \quad X_2 = \begin{bmatrix} -1 & -1 & 0 & 0 & -1 & -b & -b \\ -1 & -1 & 0 & 0 & -1 & -b & -b \\ 0 & 0 & 1 & 1 & 0 & b & -b \\ 0 & 0 & 1 & 1 & 0 & b & -b \\ -1 & -1 & 0 & 0 & -1 & -b & -b \\ -b & -b & b & b & -b & 0 & -1 \\ -b & -b & -b & -b & -b & -1 & 0 \end{bmatrix}$$

$$X_3 = \begin{bmatrix} -1 & 1 & 0 & 0 & -1 & b & b \\ 1 & -1 & 0 & 0 & 1 & -b & -b \\ 0 & 0 & 1 & -1 & 0 & b & -b \\ 0 & 0 & -1 & 1 & 0 & -b & b \\ -1 & 1 & 0 & 0 & -1 & b & b \\ b & -b & b & -b & b & 0 & -1 \\ b & -b & -b & b & b & -1 & 0 \end{bmatrix}, \quad X_4 = \begin{bmatrix} -1 & 0 & 1 & 0 & -1 & b & -b \\ 0 & 1 & 0 & -1 & 0 & b & b \\ 1 & 0 & -1 & 0 & 1 & -b & b \\ 0 & -1 & 0 & 1 & 0 & -b & -b \\ -1 & 0 & 1 & 0 & -1 & b & -b \\ b & b & -b & -b & b & 0 & 1 \\ -b & b & b & -b & -b & 1 & 0 \end{bmatrix}.$$

Then for A_1, A_2, A_3, there exists a nonzero $P \in \mathcal{S}$ such that $A_i + \delta P \in \text{PSD}_7$ for sufficiently small $\delta > 0$. For A_4, the zero matrix is the unique element P in $\mathcal{S}$ such that $A_4 + P$ is positive semi-definite.

To see the above conclusion, we use the algorithm in the last section. Clearly, we can let $V = I_7$ be the orthogonal matrix in the algorithm.

Suppose $A = A_1$. Applying Step 2 of the algorithm with $V_2 = [e_7]$, we see that $k = \dim \mathcal{S} = 4 > 2 = \dim\{V_2^t X_j : j = 1, 2, 3, 4\}$. So, there is non-zero $P \in \mathcal{S}$ such that $A + \delta P \in \text{PSD}_7$ for sufficiently small $\delta > 0$. In fact, if P is a linear combination of $X_1 + X_2$ and $X_3 + X_4$, then for sufficiently small $\delta > 0$, $A + \delta P \in \text{PSD}_7$.

Suppose $A = A_2$. Applying Step 2 of the algorithm with $V_2 = [e_6 \,|\, e_7]$, we see that $k = \dim \mathcal{S} = 4$ and since $V_2^t(X_1 + X_2 + X_3 + X_4) = 0$, $k > \dim\{V_2^t X_j : j = 1, 2, 3, 4\}$. So, there is non-zero $P \in \mathcal{S}$ such that $A + \delta P \in \text{PSD}_7$ for sufficiently small $\delta > 0$. In fact, this is true for $\delta \in [-1/4, 1/8]$ and

$$P = \sum_{i=1}^{4} X_i = \begin{bmatrix} -4 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \tag{5}$$

Suppose $A = A_3$. Applying Step 2 of the algorithm with $V_2 = [e_5 \,|\, e_6 \,|\, e_7]$, we see that $k = l = 4$; we proceed to step 3. If P is defined as in (5), $Q = \alpha A - \frac{1}{4}P \in \mathrm{PSD}_7$ where $\alpha \geq 1$. Thus, we get the desired conclusion on A_3.

Note that one can also use standard semi-definite programming packages to draw our conclusion in Step 3. To do that we consider the following optimization problem:

$$\text{Minimize / Maximize } (C, Q) \qquad \text{subject to} \quad (B_i, Q) = b_i \ \text{ and } \ Q \in \mathrm{PSD}_n.$$

Since we are interested only in feasibility, we can set C to be the zero matrix. To ensure that $Q = a_0 A + a_1 X_1 + \cdots + a_4 X_4 \in \mathrm{PSD}_n$, we set the matrices $\{B_i\}$, for $i = 1, \ldots, m$, to be a basis of $(\mathcal{S} \cup \{A\})^\perp$ in $\mathbf{S}_7$ and set $b_i = 0$. Then set $B_{m+1} = 0_4 \oplus I_3$ with $b_{m+1} = 1$. We will get the desired conclusion by running any standard semi-definite programming package.

Suppose $A = A_4 \in \mathrm{PSD}_7$. Applying Step 2 of the algorithm with $V_2 = [e_4 \,|\, e_5 \,|\, e_6 \,|\, e_7]$, we see that $k = l = 4$; we proceed to step 3. Since I_4 is orthogonal to all matrices in $\tilde{\mathcal{S}} = \mathrm{span}\,\{V_2^t X_j V_2 : j = 1, \ldots, 4\}$, we see that $I_4 \in \tilde{\mathcal{S}}^\perp \cap \mathrm{PD}_4$. By the theorem of alternative, $\tilde{\mathcal{S}} \cap \mathrm{PSD}_4 = \{0_4\}$. Thus, there is no matrix Q satisfying Step 3, and 0_7 is the only element P in $\mathcal{S}$ such that $A_4 + P \in \mathrm{PSD}_7$.

Actually, to get the conclusion on A_4 one can also check directly that the matrix Q in Step 3 of the algorithm does not exist by a straightforward verification or using standard semi-definite programming routines.

We can use Example 3.1 to get examples for the EDS matrix completion problem in the following. Denote by $\{E_{11}, E_{12}, \ldots, E_{nn}\}$ the standard basis for $n \times n$ real matrices.

Example 3.2 Let $A_1, A_2, A_3, A_4, X_1, X_2, X_3, X_4$ be defined as in Example 3.1. Suppose $\tilde{A}_1, \tilde{A}_2, \tilde{A}_3, \tilde{A}_4 \in \mathrm{EDS}_8$ are such that

$$\frac{-1}{2} U^t \tilde{A}_j U = A_j, \qquad j = 1, 2, 3, 4,$$

where

$$U = \frac{1}{\sqrt{8}} \begin{bmatrix} -1 & 1 & -1 & 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & -1 & 1 & 1 & -\sqrt{2} & 0 \\ 1 & 1 & 1 & 1 & 1 & \sqrt{2} & 0 \\ -1 & -1 & 1 & 1 & -1 & 0 & -\sqrt{2} \\ 1 & 1 & 1 & -1 & -1 & -\sqrt{2} & 0 \\ 1 & -1 & -1 & -1 & -1 & \sqrt{2} & 0 \\ -1 & 1 & -1 & -1 & 1 & 0 & -\sqrt{2} \\ -1 & -1 & 1 & -1 & 1 & 0 & \sqrt{2} \end{bmatrix}.$$

Note that the matrices $\tilde{A}_1, \ldots, \tilde{A}_4$ are determined uniquely by the result in [4]. Let $\mathcal{S} = \mathrm{span}\,\{E_{12} + E_{21}, E_{13} + E_{31}, E_{24} + E_{42}, E_{34} + E_{43}\}$. Then

$$\frac{-1}{2} U^t (E_{13} + E_{31}) U = \frac{-1}{8} X_1, \quad \frac{-1}{2} U^t (E_{34} + E_{43}) U = \frac{-1}{8} X_2,$$

$$\frac{-1}{2}U^t(E_{12}+E_{21})U = \frac{-1}{8}X_3, \quad \frac{-1}{2}U^t(E_{24}+E_{42})U = \frac{-1}{8}X_4.$$

By Proposition 2.7 and Example 3.1, we see that there exists a nonzero $P \in \mathcal{S}$ such that $\tilde{A}_j + P \in \mathrm{EDS}_8$ for $j = 1, 2, 3$, and 0_8 is the unique element P in $\mathcal{S}$ such that $\tilde{A}_4 + P \in \mathrm{EDS}_8$.

4 Comparison with the results of Alfakih

We reformulate Example 3.2 in the standard Euclidean distance squared completion problem setting and show that some results in [1] are incorrect in the following.

Continue to assume that $A_1, A_2, A_3, A_4, X_1, X_2, X_3, X_4$ are defined as in Example 3.1. Let $\tilde{A}_0$ be the partially specified matrix

$$\tilde{A}_0 = \begin{bmatrix} 0 & ? & ? & 1 & 7/4 & 7/4 & 1 & 2 \\ ? & 0 & 2 & ? & 2 & 2 & 7/4 & 7/4 \\ ? & 2 & 0 & ? & 2 & 2 & 7/4 & 7/4 \\ 1 & ? & ? & 0 & 7/4 & 7/4 & 2 & 1 \\ 7/4 & 2 & 2 & 7/4 & 0 & 2 & 7/4 & 7/4 \\ 7/4 & 2 & 2 & 7/4 & 2 & 0 & 7/4 & 7/4 \\ 1 & 7/4 & 7/4 & 2 & 7/4 & 7/4 & 0 & 1 \\ 2 & 7/4 & 7/4 & 1 & 7/4 & 7/4 & 1 & 0 \end{bmatrix}.$$

We can complete $\tilde{A}_0$ to $\tilde{A}_1$ by setting all unspecified entries to 7/4. So, we have $\frac{-1}{2}U^t\tilde{A}_1U = A_1$. If P is a linear combination of $E_{13}+E_{31}+E_{34}+E_{43}$ and $E_{12}+E_{21}+E_{24}+E_{42}$, then $\tilde{A}_1 + \delta P \in \mathrm{EDS}_8$ for sufficiently small $\delta > 0$.

Let

$$Y = \begin{bmatrix} 0 & 1/4 & 1/4 & 1 & 1/4 & 1/4 & 1 & 0 \\ 1/4 & 0 & 0 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 0 & 0 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1 & 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 & 1 \\ 1/4 & 0 & 0 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1/4 & 0 & 0 & 1/4 & 0 & 0 & 1/4 & 1/4 \\ 1 & 1/4 & 1/4 & 0 & 1/4 & 1/4 & 0 & 1 \\ 0 & 1/4 & 1/4 & 1 & 1/4 & 1/4 & 1 & 0 \end{bmatrix}.$$

Then $-U^tYU/2 = 0_6 \oplus [1]$. Furthermore, for any $X \in \mathcal{S}$, we have $(U^tXU)_{77} = 0$ and hence $(U^tYU, U^tXU) = 0$. So, [1, Theorem 3.3(2.a)] asserts that $\tilde{A}_1$ is a unique completion of $\tilde{A}_0$, which is not true by Example 3.2.

For easy comparison, we note that in the notation of [1], $\overline{r} = n - 1 - \mathrm{rank}\,(A_1) = 1$ and the Gale matrix corresponding to $\tilde{A}_1$ is

$$Z^t = \begin{bmatrix} z^1 & z^2 & z^3 & z^4 & z^5 & z^6 & z^7 & z^8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

If $\Psi = [1]$ then for $(i,j) \in \{(1,2),(1,3),(2,4),(3,4)\}$ such that the entries $(\tilde{A}_0)_{i,j}$ are free, we have $z_i^t\Psi z_j = 0$, and thus [1, Theorem 3.3 (2.a)] applies.

The construction of our example does not depend on the degeneracy that $\overline{r} = 1$. We can use a similar technique to construct an example for $\overline{r} = 2$. Let $Y \in \mathbf{S}_8^0$ be such that $-\frac{1}{2}U^tYU = 0_5 \oplus I_2$ and let B be a partial matrix that completes to $\tilde{A}_2$ with free entries in the same position as free entries of $\tilde{A}_0$. As $(U^tYU, U^tXU) = 0$ for any $X \in \mathcal{S}$, [1, Theorem 3.3(2.b)] asserts that $\tilde{A}_2$ is a unique completion of B, but this contradicts Example 3.2 which shows that the positive semi-definite completions of A_2 is not unique.

Again, for easy comparison we note that in the notation of [1], $\overline{r} = 2$ and

$$Z^t = \begin{bmatrix} z^1 & z^2 & z^3 & z^4 & z^5 & z^6 & z^7 & z^8 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & -1 & 0 & 0 & -1 & 1 \\ 0 & -1 & 1 & 0 & -1 & 1 & 0 & 0 \end{bmatrix}$$

is the Gale matrix for $\tilde{A}_2$. Let $\Psi = I_2$ and for $(i,j) \in \{(1,2),(1,3),(2,4),(3,4)\}$, the entries $(B)_{i,j}$ are free. One readily checks that for these pair, $z_i^t \Psi z_j = 0$.

Discrepancies in the proofs of Alfakih

The flaw in [1, Theorem 3.3] lies in the proof of Theorem 3.2 (Corollary 4.1) in the paper. Following the notation in [1, p. 8], we let

$$\mathcal{L} = \left\{ -U^tXU/2 : X \in \mathcal{S} \right\}, \quad \mathcal{L}^{\perp} = \left\{ X \in \mathbf{S}_{n-1} : (X,Z) = 0 \text{ for all } Z \in \mathcal{L} \right\},$$

$$K = \left\{ B \in \mathbf{S}_{n-1} : B = \lambda \left(X + \frac{1}{2} U^t \tilde{A}_0 U \right), \lambda \geq 0, X \in \mathrm{PSD}_{n-1} \right\}$$

and

$$\mathrm{int}\,(K^{\circ}) = \left\{ C \in \mathbf{S}_{n-1} : (C,B) < 0 \text{ for all } B \in K \right\}.$$

The author of [1] claimed that: *if there exists some* $Y \in \mathrm{PD}_{n-1-r}$ *such that*

$$\hat{Y} = \begin{bmatrix} 0 & 0 \\ 0 & Y \end{bmatrix} \in \mathcal{L}^{\perp}, \tag{6}$$

then

$$\mathcal{L}^{\perp} \cap \mathrm{int}(K^{\circ}) \neq \emptyset, \tag{7}$$

and hence $\mathcal{L} \cap \mathrm{cl}(K) = \{0\}$ by the theorem of alternative. However, in Example 3.2, in spite of the existence of $\hat{Y} \in \mathcal{L}^{\perp}$ of the form (6) one can check that (7) does not hold. In particular, $\hat{Y} \notin \mathrm{int}(K^{\circ})$ because $I_r \oplus 0_{n-1-r} \in K$ but $(\hat{Y}, I_r \oplus 0_{n-1-r}) = 0$.

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