NORM PRESERVERS OF JORDAN PRODUCTS*

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Abstract. Norm preserver maps of Jordan product on the algebra M_n of $n \times n$ complex matrices are studied, with respect to various norms. A description of such surjective maps with respect to the Frobenius norm is obtained: Up to a suitable scaling and unitary similarity, they are given by one of the four standard maps (identity, transposition, complex conjugation, and conjugate transposition) on M_n, except for a set of normal matrices; on the exceptional set they are given by another standard map. For many other norms, it is proved that, after a suitable reduction, norm preserver maps of Jordan product transform every normal matrix to its scalar multiple, or to a scalar multiple of its conjugate transpose.

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1. Introduction.

Let M_n be the algebra of $n \times n$ complex matrices. We denote by $X \circ Y$ the *Jordan product*: $X \circ Y = XY + YX$ for $X, Y \in M_n$.

In this paper we address the following general problem:

Problem 1.1. *Suppose $\|\cdot\|$ is a norm on M_n. Characterize the norm preservers of Jordan product, i.e. all maps $f : M_n \to M_n$ with the property that*

$$\|f(A) \circ f(B)\| = \|A \circ B\|$$

for all $A, B \in M_n$.

Recently, preserver problems with respect to various algebraic operations on M_n, including Jordan products, Jordan triple products and Lie products, attracted a lot of attention of researchers in the field; we mention [4, 5] and [1] where certain Jordan product preservers and Jordan triple product preservers, respectively, are studied. *Norm* preserver problems for Lie products are studied in [7].

Note that the maps f in Problem 1.1 are not assumed linear or continuous. In fact, our results show that there are many discontinuous norm preservers of Jordan product. Perhaps because of this circumstance, a complete solution of Problem 1.1

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for large class of norms, even well-behaved ones such as unitarily invariant, seems to be out of reach at present.

From now on we assume that $n \geq 3$ *throughout the rest of the paper.*

In this paper, we focus on *surjective* norm preservers of Jordan product, and obtain their complete characterization for the Frobenius norm (Theorem 5.1). In particular, it turns out that discontinuities of such preservers (if any) are confined to a subset of normal matrices.

A key feature of surjective norm preservers of Jordan product is that they preserve the set of normal matrices, assuming the norm is sufficiently nice; moreover (after a suitable reduction) they map a normal matrix to a unimodular multiple of itself or of its conjugate transpose. We prove this property in Section 3 (Theorem 3.1). For many unitarily invariant norms, including the Schatten p-norms with $p \notin \{1, 2, \infty\}$, the second possibility when a normal matrix is mapped to a unimodular multiple of its conjugate transpose, actually does not occur (except of course when the matrix is a multiple of its conjugate transpose); see Theorem 4.1.

The following notation and terminology will be used throughout the paper: Denote by C and $\mathsf{T} \subset \mathsf{C}$ the complex field and the unit circle, respectively. A complex number z is written as $z = \mathrm{Re}\,(z) + \mathbf{i}\mathrm{Im}\,(z)$, where $\mathrm{Re}\,(z)$ and $\mathrm{Im}\,(z)$ are the real and imaginary parts of z, respectively. C^n is the vector space of complex *column vectors* of length n; and $\mathbf{e}_1, \ldots, \mathbf{e}_n$ is its standard orthonormal basis.

Let $E_{ij} := \mathbf{e}_i \mathbf{e}_j^*$, $1 \leq i, j \leq n$, be the standard basis for M_n. The $n \times n$ identity matrix is denoted I_n or I (if n is clear from context). $\mathrm{diag}\,(a_1, \ldots, a_n)$ is the diagonal matrix with $a_1, \ldots, a_n$ on the main diagonal (in this order). We let $s_1(A) \geq s_2(A) \geq \cdots \geq s_n(A)$ be the singular values of $A \in M_n$. $\mathcal{N}_n$ stands for the set of all complex $n \times n$ normal matrices. $\mathcal{T}_n$ is the set of all matrices $X \in M_n$ which are either diagonalizable (by similarity) with spectrum $\sigma(X)$ of the form $\{\lambda, -\lambda\}$ for some $\lambda \in \mathsf{C}$, or of rank one.

A norm $\| \cdot \|$ on M_n is *unitary invariant* (UI) whenever $\|UAV\| = \|A\|$ for all unitary U and V, and all matrices $A \in M_n$. It is *unitary similarity invariant* (USI) whenever $\|UAU^*\| = \|A\|$ for all unitary U and all $A \in M_n$.

The following four *standard* bijective maps on M_n will be used:

$$X \mapsto X \text{ identity map}, \quad X \mapsto \overline{X} \text{ complex conjugation}, \tag{1.1}$$

$$X \mapsto X^{\mathrm{tr}} \text{ transposition}, \quad X \mapsto X^* \text{ conjugate transposition}. \tag{1.2}$$

2. Surjective norm preservers and the test set. In this section we show that surjective norm preservers, with respect to a large class of USI norms, are well behaved on the *test set* of matrices $\mathcal{T}_n$ (defined in the introduction).

Our main result in this section is:

Theorem 2.1. *Assume* $\| \cdot \|$ *is a USI norm such that*

$$\|Z^{\#}\| = \|Z\|, \qquad \forall \ Z \in M_n, \tag{2.1}$$

where $^{\#}$ *stands for any one of the four standard bijective maps on* M_n.

Let $f : M_n \to M_n$ be a surjective map with the property

$$\|A \circ B\| = \|f(A) \circ f(B)\| \quad \text{for all } A, B \in M_n. \tag{2.2}$$

Then there exist:

(1) *a unitary matrix T;*

(2) *a map $\gamma : \mathcal{T}_n \to \mathsf{T}$;*

(3) *a standard bijective map $^{\#}$;*

such that

$$f(X) = \gamma(X) T X^{\#} T^{*}, \qquad \forall \ X \in \mathcal{T}_n. \tag{2.3}$$

We need some preliminaries for the proof of Theorem 2.1. The starting point is the following particular case of a general result from [3].

Theorem 2.2. *Let $f : M_n \to M_n$ be a surjective map such that*

$$A \circ B = 0 \quad \Longleftrightarrow \quad f(A) \circ f(B) = 0 \quad \text{for all } A, B \in M_n. \tag{2.4}$$

Then there exist:

(1) *an invertible matrix T;*

(2) *a map γ from M_n into the set of nonzero complex numbers;*

(3) *a field isomorphism $\phi : \mathsf{C} \to \mathsf{C}$;*

such that

$$\textit{either } f(X) = \gamma(X) T X^{\phi} T^{-1}, \ \forall \ X \in \mathcal{T}_n, \quad \textit{or } f(X) = \gamma(X) T (X^{\phi})^{\mathrm{tr}} T^{-1}, \ \forall \ X \in \mathcal{T}_n. \tag{2.5}$$

Here, X^{ϕ} is obtained by entrywise application of ϕ to the entries of X.

We record the following simple lemma:

Lemma 2.3. *Let $X \mapsto X^{\#}$ be one of the four standard bijective maps. If X is a rank one matrix with real trace, then $X^{\#}$ is unitarily similar to X.*

Proof. It is easy to see that every rank one matrix X is unitarily similar to

$$Y_1 := (\operatorname{trace} X) E_{11} + \sqrt{\operatorname{trace}(X^* X) - |\operatorname{trace} X|^2}\, E_{12},$$

as well as to

$$Y_2 := (\operatorname{trace} X) E_{11} + \sqrt{\operatorname{trace}(X^* X) - |\operatorname{trace} X|^2}\, E_{21}.$$

Note that $\operatorname{trace}(X^* X) \geq |\operatorname{trace} X|^2$. If the trace is real, we obviously have $Y_1^{\#}$ equal to either Y_1 or Y_2, and we are done. ☐

Lemma 2.4. *Let $\|\cdot\|$ be a USI norm on M_n. Suppose a not necessarily surjective map $f : M_n \to M_n$ satisfies* (2.5) *(with $\gamma(X)$, ϕ and T as in Theorem* 2.2*) and $\|A \circ B\| = \|f(A) \circ f(B)\|$ for all $A, B \in \mathcal{T}_n$. Then, ϕ is either trivial or the complex conjugation. Moreover, the matrix T, which is defined up to a nonzero scalar multiple, can be chosen to be unitary, and $|\gamma(X)| = 1$ for each $X \in \mathcal{T}_n$.*

Proof. We proceed in three steps.

Step 1. ϕ is continuous, and $\gamma_0 := |\gamma(E_{ij})|$ is independent of (i,j).

To see this, fix distinct indices i,j. Given $z \in \mathsf{C}$, consider

$$A_z = E_{ii} + zE_{ij}; \quad B := E_{ij}; \quad C := E_{jj}.$$

Then

$$A_z \circ B = B, \quad A_z \circ C = zB,$$

and, assuming for example that the first alternative of (2.5) applies, we obtain

$$\begin{aligned}\|B\| = \|A_z \circ B\| = \|f(A_z) \circ f(B)\| &= |\gamma(A_z)||\gamma(B)| \cdot \|T((A_z)^{\phi} \circ B^{\phi})T^{-1}\| \\ &= |\gamma(A_z)||\gamma(B)| \cdot \|T((A_z) \circ B)^{\phi}T^{-1}\| \\ &= |\gamma(A_z)||\gamma(B)| \cdot \|TB^{\phi}T^{-1}\|,\end{aligned} \tag{2.6}$$

and analogously

$$\begin{aligned}|z| \cdot \|B\| = \|A_z \circ C\| = \|f(A_z) \circ f(C)\| &= |\gamma(A_z)||\gamma(C)| \cdot \|T((A_z)^{\phi} \circ C^{\phi})T^{-1}\| \\ &= |\gamma(A_z)||\gamma(C)| \cdot \|T(zB)^{\phi}T^{-1}\| = |\gamma(A_z)||\gamma(C)||\phi(z)| \cdot \|TB^{\phi}T^{-1}\|.\end{aligned} \tag{2.7}$$

Comparing (2.7) and (2.6), we see that

$$|\phi(z)| = |z||\gamma(B)||\gamma(C)|^{-1}. \tag{2.8}$$

It follows (using the property $\phi(z_1 - z_2) = \phi(z_1) - \phi(z_2)$ for all $z_1, z_2 \in \mathsf{C}$) that ϕ is continuous, therefore as is well known, ϕ is either trivial or the complex conjugation. Then, however, $|\phi(z)| = |z|$, so Eq. (2.8) gives $|\gamma(E_{ij})| = |\gamma(E_{jj})|$. Note that $(X \circ Y)^{\mathrm{tr}} = X^{\mathrm{tr}} \circ Y^{\mathrm{tr}}$. Hence, we may repeat the arguments with $(A_z^{\mathrm{tr}}, B^{\mathrm{tr}}, C^{\mathrm{tr}})$ in place of (A,B,C) to get $|\gamma(E_{ji})| = |\gamma(E_{jj})|$. By the arbitrariness of $i \neq j$, $|\gamma(E_{ij})|$ is constant.

We proceed similarly if the second alternative of (2.5) applies.

Step 2. We just saw that there exists a standard bijective map $X \mapsto X^{\#}$ such that

$$f(X) = \gamma(X)TX^{\#}T^{-1}, \quad \forall\, X \in \mathcal{T}_n, \tag{2.9}$$

where γ maps into $\mathsf{C} \setminus \{0\}$. We show next that T is a scalar multiple of a unitary matrix, and therefore can be chosen to be unitary.

To verify this, let $T = UDV$ be a singular value decomposition. That is, U, V are unitary, and $D := \operatorname{diag}(s_1, \ldots, s_n)$, where $s_j = s_j(T)$, $j = 1, 2, \ldots, n$.

Consider the map $\widehat{f} : X \mapsto U^* f((V^{\#})^* X V^{\#})U$ if $^{\#}$ is the identity or conjugation, or the map $\widehat{f} : X \mapsto U^* f(V^{\#} X (V^{\#})^*)U$ if $^{\#}$ is the transposition or conjugate transposition. Since the norm is USI, $\|A \circ B\| = \|\widehat{f}(A) \circ \widehat{f}(B)\|$ for all $A, B \in \mathcal{T}_n$ remains valid. Moreover, a computation shows that

$$\widehat{f}(X) = \widehat{\gamma}(X) D X^{\#} D^{-1}, \quad \forall\, X \in \mathcal{T}_n,$$

where $\widehat{\gamma}$ maps M_n into $\mathsf{C} \setminus \{0\}$. If we could now infer $D = \lambda I$ then $T = \lambda UV$ will indeed be a multiple of unitary.

To this end, note that $E_{ii} \circ E_{ij} = E_{ij}$ for $i \neq j$. By the first step, applied to $\widehat{f}$, $\widehat{\gamma}_0 := |\widehat{\gamma}(E_{ij})|$ is independent of (i, j). Therefore, for $i \neq j$ we have

$$\begin{aligned}\|E_{ij}\| = \|E_{ii} \circ E_{ij}\| = \|\widehat{f}(E_{ii}) \circ \widehat{f}(E_{ij})\| &= \widehat{\gamma}_0^2 \|D(E_{ii}^{\#} \circ E_{ij}^{\#})D^{-1}\| \\ = \widehat{\gamma}_0^2 \|DE_{ij}^{\#}D^{-1}\| &= \widehat{\gamma}_0^2 (s_i s_j^{-1})^{\pm 1} \|E_{ij}^{\#}\|,\end{aligned}$$

where the sign ± 1 depends on the standard bijective map ${}^{\#}$. However, by Lemma 2.3 a rank-one $E_{ij}^{\#}$ is unitarily similar to E_{ij}, so that $\|E_{ij}^{\#}\| = \|E_{ij}\|$. We deduce $s_i s_j^{-1}$ is the same for all $i \neq j$. It follows that $s_1 = \cdots = s_n$, so D is scalar.

Step 3. $|\gamma(X)| = 1$ for all $X \in \mathcal{T}_n$.

In fact, by the second step, $T = U$ is unitary and $f(X) = \gamma(X) U X^{\#} U^*$. We may assume $U = I$ in the sequel, otherwise, replace f by a mapping $X \mapsto U^* f(X) U$.

Now, to demonstrate $|\gamma(X)| = 1$, suppose first $X = \lambda P \in \mathcal{T}_n \setminus \{0\}$ is a scalar multiple of a rank-one idempotent P. Then,

$$2|\lambda|^2 \cdot \|P\| = \|X \circ X\| = \|f(X) \circ f(X)\| = |\gamma(X)|^2 \cdot \|(2\lambda^2 P)^{\#}\|.$$

By Lemma 2.3, a rank-one idempotent $P^{\#}$ is unitarily similar to P. Therefore,

$$\|(2\lambda^2 P)^{\#}\| = |2\lambda^{\#}|^2 \cdot \|P\| = 2|\lambda|^2 \cdot \|P\|.$$

Since $\lambda \neq 0$, we deduce $|\gamma(X)| = 1$.

Suppose next X is a rank-one nilpotent. Then, there exists a unitary U such that $X = zUE_{12}U^*$, $z \in \mathsf{C} \setminus \{0\}$. Let $P := U(E_{11} + E_{12})U^*$ be a rank-one idempotent. We have $P \circ X = X$, so that

$$\begin{aligned}\|X\| = \|P \circ X\| = \|f(P) \circ f(X)\| &= |\gamma(P)|\,|\gamma(X)| \cdot \|P^{\#} \circ X^{\#}\| \\ = 1 \cdot |\gamma(X)| \cdot \|(P \circ X)^{\#}\| &= |\gamma(X)| \cdot \|X^{\#}\|.\end{aligned}$$

By Lemma 2.3, $X^{\#}$ is unitarily similar to X. Hence, $|\gamma(X)| = 1$.

Lastly, suppose X is a nonzero diagonalizable matrix with the spectrum equal to $\{-\lambda, \lambda\}$. Then, $X^2 = \lambda^2 I$, so

$$2|\lambda^2| \|I\| = \|X \circ X\| = \|f(X) \circ f(X)\| = |\gamma(X)|^2 \cdot \|X^{\#} \circ X^{\#}\| = 2|\gamma(X)|^2 |\lambda^2| \cdot \|I\|.$$

Yet again we deduce $|\gamma(X)| = 1$. Finally, if $X = 0$, then obviously $f(X) = 0$, and we can take $|\gamma(X)| = 1$ as well. □

Proof of Theorem 2.1. The proof follows by combining Theorem 2.2 and Lemma 2.4. □

3. Reduced maps. We say that a map $f : M_n \to M_n$ is *reduced* (with respect to the norm $\|\cdot\|$), notation $f \in \mathcal{R}_n$, if the following properties are satisfied:

(1) $\|A \circ B\| = \|f(A) \circ f(B)\|$ for all $A, B \in M_n$;

(2) $f(X) = \gamma(X)X$ for every $X \in \mathcal{T}_n$, where $\gamma(X) \in \mathsf{T}$.

In view of Theorem 2.1, after a suitable unitary similarity transformation, there is no essential loss of generality in assuming that our surjective norm preservers of Jordan product are reduced (if the norm satisfies the hypotheses of Theorem 2.1).

The main result in this section asserts that, under appropriate hypotheses on the norm, reduced maps transform a normal matrix to either its unimodular scalar multiple, or to a unimodular scalar multiple of its conjugate transpose:

Theorem 3.1. *Assume that $\|\cdot\|$ is a USI norm with the following properties:*

(a) *$\|Z^{\#}\| = \|Z\|$ for every $Z \in M_n$ and for every standard bijective map $^{\#}$.*

(b) *For every block diagonal matrix $B_1 \oplus B_2$, we have $\|B_1 \oplus B_2\| = \|B_1 \oplus (-B_2)\|$.*

(c) *Strict convexity: The equality*

$$\|X + Y\| = \|X\| + \|Y\|, \qquad X, Y \in M_n$$

implies that one of X and Y is a nonnegative multiple of the other.

Let f be any reduced map with respect to $\|\cdot\|$. Then for every normal matrix $A \in M_n$ we have

$$f(A) = \mu A \qquad \text{or} \qquad f(A) = \mu A^*, \tag{3.1}$$

where $\mu = \mu(A) \in \mathsf{T}$.

We need a few lemmas for the proof.

The first is an elementary fact on complex numbers.

Lemma 3.2. *Let $a_1, a_2, \ldots, a_n$ and $b_1, b_2, \ldots, b_n$, $n \geq 2$, be complex numbers such that*

$$|a_i| = |b_i|, \quad i = 1, 2, \ldots, n, \tag{3.2}$$

$$|a_i + a_j| = |b_i + b_j|, \quad j \neq i, \quad i, j = 1, 2, \ldots, n. \tag{3.3}$$

Then there exists a $\mu \in \mathsf{T}$ such that at least one of the following two possibilities holds:

(1) $(a_1, a_2, \ldots, a_n) = \mu (b_1, b_2, \ldots, b_n)$; (2) $(a_1, a_2, \ldots, a_n) = \mu \left(\overline{b_1}, \overline{b_2}, \ldots, \overline{b_n}\right)$.

Proof. Without loss of generality, we assume that all $a_1, a_2, \ldots, a_n$ as well as $b_1, b_2, \ldots, b_n$, $n \geq 2$, are all nonzero. We prove the result by induction on n.

Let first $n = 2$. Scaling the given numbers, $a_j \to \alpha a_j$, $b_j \to \beta b_j$, $j = 1, 2$, where $\alpha, \beta \in \mathbb{C}$ are such that $|\alpha| = |\beta| \neq 0$, we may further assume that $a_1 = b_1 = 1$. Then, $|a_2| = |b_2|$ together with $|1 + a_2| = |1 + b_2|$ imply that a_2 and b_2 lie in the intersection of two circles, centered at origin and at -1, respectively, whence $a_2 = b_2$ or $a_2 = \overline{b_2}$.

Assume now $n \geq 3$ and that the Lemma holds true for any $2 \leq k < n$. Let a_j, b_j satisfy (3.2), (3.3). We may assume that the a_j's and b_j's are all nonzero. By induction hypothesis and replacing $b_1, b_2, \ldots, b_n$ with $\overline{b_1}, \overline{b_2}, \ldots, \overline{b_n}$, if necessary, we may assume that there exists a $\mu \in \mathsf{T}$ such that

$$(a_1, a_2, \ldots, a_{n-1}) = \mu (b_1, b_2, \ldots, b_{n-1}). \tag{3.4}$$

Applying induction hypothesis again, we have (a) $(a_2,\ldots,a_n)=\mu'(b_2,\ldots,b_n)$ or (b) $(a_2,\ldots,a_n)=\mu'\left(\overline{b_2},\ldots,\overline{b_n}\right)$ for some $\mu'\in\mathsf{T}$. If (a) holds true, from $\mu b_2=\mu' b_2$ follows that $\mu=\mu'$ and we are done. If (b) is satisfied, then there is a $\mu''\in\mathsf{T}$ such that either $(a_1,a_n)=\mu''(b_1,b_n)$ or $(a_1,a_n)=\mu''\left(\overline{b_1},\overline{b_n}\right)$. The first possibility, together with (3.4) implies that $\mu''=\mu$, and consequently, $a_n=\mu'' b_n=\mu b_n$ yields $(a_1,a_2,\ldots,a_n)=\mu(b_1,b_2,\ldots,b_n)$. The case $(a_1,a_n)=\mu''\left(\overline{b_1},\overline{b_n}\right)$ gives that $\mu''=\mu'$, hence $a_1=\mu''\overline{b_1}=\mu'\overline{b_1}$ and $(a_1,a_2,\ldots,a_n)=\mu'\left(\overline{b_1},\overline{b_2},\ldots,\overline{b_n}\right)$, which completes the proof. □

Note that every USI norm has the block monotonicity property:

Lemma 3.3. *The following inequality holds:*

$$\left\|\begin{bmatrix} A & B\\ C & D\end{bmatrix}\right\| \ge \left\|\begin{bmatrix} A & 0\\ 0 & D\end{bmatrix}\right\|,$$

where A, B, C, and D are arbitrary blocks of sizes $k\times k$, $k\times(n-k)$, $(n-k)\times k$, and $(n-k)\times(n-k)$, respectively.

Proof. Letting $P=I_k\oplus -I_{n-k}$, note that $X:=\begin{bmatrix} A & B\\ C & D\end{bmatrix}$ and $Y:=PXP$ are unitarily similar and hence $\|X\|=\|Y\|$. Then

$$Z:=\begin{bmatrix} A & 0\\ 0 & D\end{bmatrix}=\frac{1}{2}(X+Y)$$

and $\|Z\|\le(\|X\|+\|Y\|)/2=\|X\|$. □

Lemma 3.4. *Suppose $\|\cdot\|$ is a USI norm. Then the equality $\|I\circ X\|=\|B\circ X\|$ for all rank one $X\in M_n$ implies that $B=\mu I$ with $\mu\in\mathsf{T}$.*

Proof. Let U be unitary such that $U^*BU=[b_{ij}]_{i,j=1}^n$ is in upper triangular form. Then for $X=UE_{1n}U^*$ we have

$$2\|E_{1n}\|=\|I\circ X\|=\|B\circ X\|=\|U^*BU\circ E_{1n}\|=\|(b_{11}+b_{nn})E_{1n}\|.$$

Thus, $|b_{11}+b_{nn}|=2$. If $b_{11}\neq b_{nn}$ then, by the triangle inequality, $|b_{11}|>1$ or $|b_{nn}|>1$. If $|b_{nn}|>1$, then for $Y=UE_{nn}U^*$, we have

$$2\|E_{nn}\|=\|I\circ Y\|=\|B\circ Y\|=\|U^*BU\circ E_{nn}\|\ge 2|b_{nn}|\|E_{nn}\|. \tag{3.5}$$

In the last inequality, Lemma 3.3 was used. But (3.5) clearly contradicts $|b_{nn}|>1$. Analogously, we prove that $|b_{11}|>1$ is impossible. Thus, $b_{11}=b_{nn}$ has modulus 1. Since this is true for any unitary U such that U^*BU is in triangular form, and since the eigenvalues of B can be arranged in any prescribed order on the main diagonal of the upper triangular matrix U^*BU (for a suitable unitary U), we see that B has all eigenvalues equal to μ with modulus 1. We may replace B by B/μ and assume that $\mu=1$.

If $B\neq I$, there is a unitary U such that U^*BU has $(1,1)$ entry equal to $d>1$ (this is easily seen by considering the numerical range of B). Then for $Z_1:=U^*BU\circ E_{11}$

and $Z_2 := (I-2E_{11})Z_1(I-2E_{11})$, we have $\|Z_1\| = \|Z_2\|$ (because $I-2E_{11}$ is unitary and Hermitian), and $(Z_1+Z_2)/2 = 2dE_{11}$. Thus,

$$\begin{aligned} 2\|E_{11}\| &= \|I \circ UE_{11}U^*\| = \|B \circ UE_{11}U^*\| = \|Z_1\| \\ &= \|Z_1\|/2 + \|Z_2\|/2 \geq \|(Z_1+Z_2)/2\| = 2d\|E_{11}\|, \end{aligned}$$

a contradiction. So, $B = I$. □

Lemma 3.5. *Assume $\|\cdot\|$ is a USI norm. If f is a reduced map with respect to $\|\cdot\|$, and if P is a Hermitian projection, then $f(P) = \gamma P$ for some $\gamma \in \mathsf{T}$.*

Proof. For simplicity, assume $P = 0_k \oplus I_{n-k}$ (the general case is easily reduced to this one). If $n-k = 1$ or if $n-k = 0$ (i.e., $P = 0$) there is nothing to do: the result follows from the definition of reduced maps. So, let $n-k \geq 2$. From $P \circ E_{ii} = 0$, $i = 1, 2, \ldots, k$, it follows that $B := f(P) = 0_k \oplus B_1$ for some $B_1 \in M_{n-k}$. For any rank-one $(n-k) \times (n-k)$ matrix X we have

$$\|P \circ (0_k \oplus X)\| = \|B \circ (0_k \oplus X)\|$$

and consequently,

$$2\,\|X\| = \|B_1 \circ X\|\,, \tag{3.6}$$

where $\|\cdot\|$ in (3.6) is the norm on M_{n-k} induced by the original norm $\|\cdot\|$ on M_n:

$$\|Z\| = \left\| \begin{bmatrix} 0 & 0 \\ 0 & Z \end{bmatrix} \right\|, \quad Z \in M_{n-k}.$$

Lemma 3.4 gives that $B_1 = \gamma I$, $|\gamma| = 1$. □

Proof of Theorem 3.1. Assume the hypotheses of Theorem 3.1, and let $A \in M_n$ be normal. Let $\mathbf{g}_1, \ldots, \mathbf{g}_n$ be an orthonormal basis of eigenvectors of A; thus, $A\mathbf{g}_j = a_j\mathbf{g}_j$ for $j = 1, 2, \ldots, n$, where $a_1, \ldots, a_n$ are the corresponding eigenvalues. Denote $B := f(A)$. By Lemma 3.5, $f(I) = \gamma I$ for some $\gamma \in \mathsf{T}$, therefore we have

$$2\|A\| = \|I \circ A\| = \|f(I) \circ f(A)\| = \|I \circ B\| = 2\|B\|. \tag{3.7}$$

Consider next

$$X = -\mathbf{g}_1\mathbf{g}_1^* + \mathbf{g}_2\mathbf{g}_2^* + \cdots + \mathbf{g}_n\mathbf{g}_n^*.$$

Clearly,

$$A \circ X = -2a_1\mathbf{g}_1\mathbf{g}_1^* + 2a_2\mathbf{g}_2\mathbf{g}_2^* + \cdots + 2a_n\mathbf{g}_n\mathbf{g}_n^*.$$

Therefore, by the property (b) of $\|\cdot\|$, we have

$$2\|A\| = \|A \circ X\|. \tag{3.8}$$

Since $X \in \mathcal{T}_n$, we have $f(X) = \gamma(X)X$ with $\gamma(X) \in \mathsf{T}$. It will be convenient also to introduce the unitary matrix U with the property that $U\mathbf{g}_j = \mathbf{e}_j$, $j = 1, 2, \ldots, n$, and denote

$$UBU^* = [b_{ij}]_{i,j=1}^n = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & B_{22} \end{bmatrix},$$

where $B_{22} \in M_{n-1}$, b_{12} is an $(n-1)$-component row, and b_{21} is an $(n-1)$-component column. Thus,

$$UAU^* = \operatorname{diag}(a_1, \ldots, a_n).$$

Now, using (3.7) and (3.8), as well as the property (b) of $\|\cdot\|$, we compute:

$$\begin{aligned} 2\|B\| = 2\|A\| &= \|A \circ X\| = \|B\ f(X)\| \\ &= \|B \circ X\| = \|UBU^* \circ UXU^*\| = 2\|(-b_{11}) \oplus B_{22}\| \\ &= 2\|b_{11} \oplus B_{22}\| = \left\| \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & B_{22} \end{bmatrix} + \begin{bmatrix} b_{11} & -b_{12} \\ -b_{21} & B_{22} \end{bmatrix} \right\| \\ &\leq \|UBU^*\| + \left\| \begin{bmatrix} b_{11} & -b_{12} \\ -b_{21} & B_{22} \end{bmatrix} \right\| \\ &= \|UBU^*\| + \|(I \oplus (-I))UBU^*(I \oplus (-I))^{-1}\| = 2\|B\|, \end{aligned} \tag{3.9}$$

where the last equality follows by the unitary similarity invariance of the norm. Thus, the inequality $\geq$ is actually equality in (3.9), and the strict convexity of the norm yields $b_{12} = 0$ and $b_{21} = 0$.

We may proceed analogously with $X := I - 2\mathbf{g}_j\mathbf{g}_j^*$ $(j = 2, 3, \ldots n)$ in place of $I - 2\mathbf{g}_1\mathbf{g}_1^*$ to deduce that UBU^* is diagonal, say,

$$UBU^* = \operatorname{diag}(b_1, \ldots, b_n).$$

Now consider

$$\|\operatorname{diag}(a_1, \ldots, a_n) \circ E_{jj}\| = \|A \circ U^*E_{jj}U\| =$$

$$(\text{because } U^*E_{jj}U \in \mathcal{T}_n) \quad = \|B \circ U^*E_{jj}U\| = \|\operatorname{diag}(b_1, \ldots, b_n) \circ E_{jj}\|$$

to see that $|a_j| = |b_j|$ for all $j = 1, \ldots, n$. Analogous consideration with E_{ij}, $i \neq j$, in place of E_{jj} yields $|a_i + a_j| = |b_i + b_j|$. By Lemma 3.2 it follows that either $UBU^* = \mu UAU^*$ or $UBU^* = \mu(UAU^*)^* = U(\mu A^*)U^*$, and (3.1) follows. □

4. UI norms. In this section we continue our study of reduced maps, assuming that the norm $\|\cdot\|$ is an UI norm. Note that every UI norm satisfies the properties (a) and (b) of Theorem 3.1. If f is a reduced map with respect to a strictly convex UI norm $\|\cdot\|$, then by Theorem 3.1, for every normal X either $f(X) = \gamma(X)X$ or else $f(X) = \gamma(X)X^*$, where $|\gamma(X)| = 1$. In particular, this property holds for every

X which is a scalar multiple of a unitary matrix. However, for any unitary U, any $\mu \in \mathsf{C}$, and any $Y \in M_n$ we have

$$\begin{aligned}\|\mu U \circ Y\| = \|\mu UY + \mu YU\| \; &= \; \|\mu Y + \mu U^*YU\| \\ = \|\mu YU^* + \mu U^*Y\| \; &= \; \|\overline{\mu}YU^* + \overline{\mu}U^*Y\| \; = \; \|(\mu U)^* \circ Y\|,\end{aligned}$$

(in the second and third equality the UI property of the norm was used), and

$$\|Y \circ Z\| = \|Y^* \circ Z^*\|, \qquad \forall \quad Y, Z \in M_n.$$

Thus, if f is a reduced map with respect to $\|\cdot\|$, the property of being reduced is not affected if $f(\mu U)$ is replaced with $(f(\mu U))^*$, for every pair $(\mu, U) \in \mathcal{W}$, where $\mathcal{W}$ is a fixed (perhaps empty) subset of the set

$$\{(\mu, U) \, : \, \mu \in \mathsf{C}, \quad U \in M_n \text{ is unitary}\}.$$

Also, the reduced property is not affected if $f(X)$ is replaced with $\delta(X)f(X)$, where $\delta(X) \in \mathsf{T}$ depends on $X \in M_n$. Using these replacements, starting with a given reduced map f, we may obtain a new reduced map $\widehat{f}$, with the following properties:

(a) $\|A \circ B\| = \|\widehat{f}(A) \circ \widehat{f}(B)\|$ for all $A, B \in M_n$;

(b) $\widehat{f}(X) = X$ for every $X \in \mathcal{T}_n$ and every scalar multiple of unitary $X \in M_n$.

Maps $\widehat{f}$ with the properties (a) and (b) are said to belong to the class $\mathcal{RR}_n$ (for *restricted reduced class*).

For a quite wide class of strictly convex UI norms, we will see that the a map $\widehat{f} \in \mathcal{RR}_n$ cannot send any normal A into a unimodular multiple of A^* (unless of course A^* and A are proportional). It is well known that for every UI norm $\|\cdot\|$ on M_n there exists a symmetric gauge function g such that

$$\|A\| = g\left(s_1\left(A\right), s_2\left(A\right), \ldots, s_n\left(A\right)\right), \quad A \in M_n.$$

Consider the following additional property of g:

(P) *for every positive u and nonnegative $s_3, \ldots, s_n$, the function*

$$t \mapsto h_g\left(t\right) := g\left(\sqrt{u+t}, \sqrt{u-t}, s_3, \ldots, s_n\right), \quad 0 \le t \le u,$$

is injective.

For example, let

$$\|A\|_p = \left\{\sum_{j=1}^{n} s_j(A)^p\right\}^{1/p}, \quad 1 \le p < \infty; \quad \|A\|_\infty = s_1(A), \qquad A \in M_n,$$

be the Schatten p-norm; $1 \le p \le \infty$. Thus, $\|\cdot\|_2$ is the Frobenius norm, and $\|\cdot\|_\infty$ is the operator norm. The Schatten p-norm has the property (P) if and only if $p \neq 2$, since the function

$$t \mapsto \left((u+t)^{p/2} + (u-t)^{p/2} + \sum_{i=3}^{n} s_i^p\right)^{1/p}, \qquad 0 \le t \le u,$$

is strictly monotone if $p \neq 2$. Note also that $\|\cdot\|_p$ is strictly convex (see Theorem 3.1(c)) if and only if $1 < p < \infty$. Indeed, for

$$X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

we have $\|X+Y\|_\infty = \|X\|_\infty + \|Y\|_\infty$, thus $\|\cdot\|_\infty$ is not strictly convex. As $\|A+I\|_1 = \|A\|_1 + \|I\|_1$ for any positive semidefinite matrix A, $\|\cdot\|_1$ is not strictly convex either. For the strict convexity of $\|\cdot\|_p$, $1 < p < \infty$, see for example [8].

We show in the next theorem that restricted reduced maps with respect to strictly convex UI norms having the property (P), leave normal matrices invariant (up to scaling). Thus, under these hypotheses, the second alternative in (3.1) cannot occur. In particular, the theorem is valid for all Schatten p-norms, $p \notin \{1, 2, \infty\}$.

Theorem 4.1. *Let* $\|\cdot\|$ *be a strictly convex UI norm with the property* (P). *Let* $\widehat{f} \in \mathcal{RR}_n$. *Then* $\widehat{f}(A) = \gamma(A)\, A$, *where* $\gamma(A) \in \mathsf{T}$, *for every normal* $A \in M_n$.

Proof. Fix a normal $A \neq 0$ (if $A = 0$, then $\widehat{f}(A) = 0$, and the result is trivial). There exist a unitary matrix U, and $\alpha, y_3, \ldots, y_n \in \mathbb{C}$, $x \in \mathbb{C} \setminus \{0\}$, such that $A = xU \operatorname{diag}(1, \alpha, y_3, \ldots, y_n)\, U^* = xUDU^*$. Split $D = \operatorname{diag}(1, \alpha) \oplus Y$, $Y = \operatorname{diag}(y_3, \ldots, y_n)$. If rank A is equal to one, or A is either a multiple of a unitary matrix or a unimodular multiple of A^*, there is nothing to prove because $\widehat{f} \in \mathcal{RR}_n$, and in view of Theorem 3.1. So we assume that $\alpha \notin \mathbb{R}$ and that $|\alpha| \neq 1$. We already know by Theorem 3.1 that either $\widehat{f}(A) = \gamma(A)\, A$ or $\widehat{f}(A) = \gamma(A)\, A^*$, with $\gamma(A) \in \mathsf{T}$. We show that the latter is impossible.

On the contrary, assume that A is as above and $\widehat{f}(A) = \gamma(A)\, A^*$. Then for every $X \in \mathcal{T}_n$:

$$\begin{aligned} |x|\, \|D \circ X\| = |x|\, \|UDU^* \circ UXU^*\| \;&=\; \|A \circ UXU^*\| \;=\; \left\|\widehat{f}(A) \circ \widehat{f}(UXU^*)\right\| \\ = \|\gamma(A)\, A^* \circ UXU^*\| \;&=\; |x|\, \left\|U\overline{D}U^* \circ UXU^*\right\| \;=\; |x|\, \left\|\overline{D} \circ X\right\|. \end{aligned}$$

We will find a matrix $X \in \mathcal{T}_n$ such that $\|D \circ X\| \neq \left\|\overline{D} \circ X\right\|$, thereby obtaining a contradiction.

Let $X = X_1 \oplus \lambda I_{n-2}$, $X_1 = \begin{bmatrix} 1 & r_1 \\ ir_2 & -1 \end{bmatrix}$, where $\lambda \in \sigma(X_1)$ and $r_1, r_2 > 0$. Note that $\sigma(X) = \{\lambda, -\lambda\}$ and $X \in \mathcal{T}_n$. Let us compute the singular values of matrices

$D\circ X$ and $\overline{D}\circ X$:

$$\begin{aligned}
D\circ X &= \begin{bmatrix} 2 & (1+\alpha)\,r_1 \\ ir_2\,(1+\alpha) & -2\alpha \end{bmatrix} \oplus 2\lambda Y,\\
\overline{D}\circ X &= \begin{bmatrix} 2 & (1+\overline{\alpha})\,r_1 \\ ir_2\,(1+\overline{\alpha}) & -2\overline{\alpha} \end{bmatrix} \oplus 2\lambda Y,\\
(D\circ X)\,(D\circ X)^* &= \begin{bmatrix} u\,(\alpha) & v\,(\alpha) \\ \overline{v\,(\alpha)} & w\,(\alpha) \end{bmatrix} \oplus 4\,|\lambda|^2 \operatorname{diag}\left(|y_3|^2,\ldots,|y_n|^2\right),\\
\left(\overline{D}\circ X\right)\left(\overline{D}\circ X\right)^* &= \begin{bmatrix} u\,(\overline{\alpha}) & v\,(\overline{\alpha}) \\ \overline{v\,(\overline{\alpha})} & w\,(\overline{\alpha}) \end{bmatrix} \oplus 4\,|\lambda|^2 \operatorname{diag}\left(|y_3|^2,\ldots,|y_n|^2\right),\\
u\,(\alpha) &:= 4+|1+\alpha|^2\,r_1^2 = u\,(\overline{\alpha}),\\
w\,(\alpha) &:= r_2^2\,|1+\alpha|^2+4\,|\alpha|^2 = w\,(\overline{\alpha}),\\
v\,(\alpha) &:= -2ir_2\,(1+\overline{\alpha})-2\overline{\alpha}\,(1+\alpha)\,r_1.
\end{aligned}$$

It is easy to see that we can choose $r_1, r_2 > 0$ such that $u\,(\alpha) = w\,(\alpha)$. Let s_1, s_2 and q_1, q_2 be the singular values of the upper left 2×2 corner of matrices $D\circ X$ and $\overline{D}\circ X$, respectively. Then

$$\begin{aligned}
s_1 &= (u\,(\alpha)+|v\,(\alpha)|)^{1/2},\\
s_2 &= (u\,(\alpha)-|v\,(\alpha)|)^{1/2},\\
q_1 &= (u\,(\alpha)+|v\,(\overline{\alpha})|)^{1/2},\\
q_2 &= (u\,(\alpha)-|v\,(\overline{\alpha})|)^{1/2},
\end{aligned}$$

and, denoting by g the symmetric gauge function associated with $\|\cdot\|$,

$$\begin{aligned}
\|D\circ X\| &= g\,(s_1, s_2, 2\,|\lambda|\,|y_3|,\ldots 2\,|\lambda|\,|y_n|)\\
&= g\left(\sqrt{u\,(\alpha)+|v\,(\alpha)|}, \sqrt{u\,(\alpha)-|v\,(\alpha)|}, 2\,|\lambda|\,|y_3|,\ldots 2\,|\lambda|\,|y_n|\right)\\
&= h_g\,(|v\,(\alpha)|),\\
\left\|\overline{D}\circ X\right\| &= g\,(q_1, q_2, 2\,|\lambda|\,|y_3|,\ldots 2\,|\lambda|\,|y_n|)\\
&= g\left(\sqrt{u\,(\alpha)+|v\,(\overline{\alpha})|}, \sqrt{u\,(\alpha)-|v\,(\overline{\alpha})|}, 2\,|\lambda|\,|y_3|,\ldots 2\,|\lambda|\,|y_n|\right)\\
&= h_g\,(|v\,(\overline{\alpha})|).
\end{aligned}$$

Finally, as $|\alpha|^2 \neq 1$, α not real and $r_1, r_2 > 0$, we have

$$|v\,(\alpha)|^2 - |v\,(\overline{\alpha})|^2 = 16r_1r_2\left(|\alpha|^2-1\right)\operatorname{Im}(\alpha) \neq 0,$$

which implies that $\|D\circ X\| = h_g\,(|v\,(\alpha)|) \neq h_g\,(|v\,(\overline{\alpha})|) = \left\|\overline{D}\circ X\right\|$. □

5. Frobenius norm. *Throughout Sections* 5 *and* 6, $\|\cdot\|$ *stands for the Frobenius norm.*

In this section we state our main results that provide description of all surjective Frobenius norm Jordan product preserving maps.

Theorem 5.1. *Let $f : M_n \to M_n$ be a surjective map such that*

$$\|A \circ B\| = \|f(A) \circ f(B)\| \quad \text{for all } A, B \in M_n. \tag{5.1}$$

Then there exist:

(1) *a unitary matrix W;*
(2) *a map $\gamma : M_n \to \mathsf{T}$;*
(3) *a standard map $X \mapsto X^{\#}$;*
(4) *a subset $\mathcal{N}_0$, possibly empty, of $\mathcal{N}_n$, the set of all $n \times n$ normal matrices;*

such that

$$f(X) = \begin{cases} \gamma(X) W X^{\#} W^* & \text{if } X \in M_n \setminus \mathcal{N}_0, \\ \gamma(X) W (X^{\#})^* W^* & \text{if } X \in \mathcal{N}_0. \end{cases} \tag{5.2}$$

Theorem 5.1 admits a converse statement, as follows.

Theorem 5.2. *If $f : M_n \to M_n$ is a not necessarily surjective map given by the formula* (5.2)*, subject to conditions* (1) - (4) *of Theorem* 5.1*, then f satisfies* (5.1)*.*

We relegate proofs of Theorems 5.1 and 5.2 to Section 6.

If f is assumed to be, in addition, continuous, then more can be said:

Theorem 5.3. *Let $f : M_n \to M_n$ be a continuous surjective map such that* (5.1) *holds. Then there exist:*

($1'$) *a unitary matrix W;*
($2'$) *a map $\gamma : M_n \to \mathsf{T}$ which is continuous on $M_n \setminus \{0\}$;*
($3'$) *a standard map $X \mapsto X^{\#}$;*

such that

$$f(X) = \gamma(X) W X^{\#} W^*, \quad \forall X \in M_n. \tag{5.3}$$

Conversely, if $f : M_n \to M_n$ is a not necessarily surjective map given by the formula (5.3)*, subject to conditions* ($1'$) - ($3'$)*, then f is continuous on M_n and satisfies* (5.1)*.*

Proof. By Theorem 5.1, f has the form (5.2). It is easy to see that the set $M_n \setminus \mathcal{N}_0$ is dense in M_n. Fix $X \in \mathcal{N}_0$, and let $\{X_m\}_{m=1}^{\infty} \subset M_n \setminus \mathcal{N}_0$ be a sequence such that $\lim_{m\to\infty} X_m = X$. Passing to a subsequence if necessary we may assume that $\lim_{m\to\infty} \gamma(X_m) = \gamma$ for some $\gamma \in \mathsf{T}$. Now the continuity of f implies $\gamma W X^{\#} W^* = \gamma(X) W (X^{\#})^* W^*$, and therefore in the form (5.2) we may assume that $\mathcal{N}_0 = \emptyset$. Now if the (i, j)th entry $[X^{\#}]_{ij}$ of $X^{\#}$ is nonzero, then we have

$$\gamma(X) = \frac{[W^* f(X) W]_{ij}}{[X^{\#}]_{ij}},$$

and the continuity of γ on $M_n \setminus \{0\}$ follows. The converse statement is immediate from Theorem 5.2. □

6. Proofs of Theorems 5.1 and 5.2. For the proof of Theorem 5.1 we need the following preliminary result obtained in [6, Theorem 3.2]. We denote by $\operatorname{diagv}(X) = [x_{11}, \dots, x_{nn}]^{\mathrm{tr}} \in \mathsf{C}^n$ the diagonal vector of a matrix $X = [x_{ij}]_{i,j=1}^n$.

Theorem 6.1. *Let $A, B \in M_n(\mathsf{C})$, where $n \geq 2$. Then the following three statements are equivalent:*

(i)

$$|\mathbf{x}^* A\mathbf{x}| = |\mathbf{x}^* B\mathbf{x}| \quad \text{for all} \quad \mathbf{x} \in \mathsf{C}^n. \tag{6.1}$$

(ii) *For each unitary V there exists $\gamma(V) \in \mathsf{T}$ such that*

$$\operatorname{diagv}(VBV^*) = \gamma(V) \operatorname{diagv}(VAV^*)^{h_V},$$

where $h_V : \mathsf{C} \to \mathsf{C}$ is either identity or complex conjugation (which may depend on V);

(iii) *$B = \gamma A$ or $B = \gamma A^*$ for some $\gamma \in \mathsf{T}$.*

Let f satisfy the hypotheses of Theorem 5.1. Obviously, f also satisfies the hypotheses of Theorem 2.1. So we may assume that f has the form as in Theorem 2.1, thus (2.3) holds. Neither the assumptions of Theorem 5.1, nor the end result will be affected if we replace f by $Y \mapsto (T^* f(Y) T)^{\#}$. This way, we may assume $f(X) = \gamma(X) X$ for $X \in \mathcal{T}_n$. We may further adjust f on a subset of $\mathcal{T}_n$ so that

$$f(X) = X, \quad \forall\, X \in \mathcal{T}_n. \tag{6.2}$$

We assume therefore for the rest of this section that f is a surjective map that satisfies (5.1) and (6.2).

Remark 6.2. By Lemma 3.5, $f(I) = \gamma(I) I$ for some $\gamma(I) \in \mathsf{T}$. It follows that

$$\|f(A)\| = \|f(A) \circ f(I)\|/2 = \|A \circ I\|/2 = \|A\| \qquad \forall\, A \in M_n.$$

Observe that Frobenius norm is strictly convex and satisfies properties (a)–(c) of Theorem 3.1. The following lemma therefore follows immediately from the conclusions of Theorem 3.1.

Lemma 6.3. *If $A \in M_n$ is normal, then $f(A) = \gamma(A)A$ or $f(A) = \gamma(A)A^*$ for some $\gamma(A) \in \mathsf{T}$. In particular, f maps the set $\mathcal{N}_n$ into itself.*

Lemma 6.4.

(a) *If $A \in M_n$ is normal, then $\|A \circ X\| = \|A^* \circ X\|$ for every $X \in M_n$.*

(b) *If $X \in M_n$ is normal, then $\|B \circ X\| = \|B^* \circ X\|$ for every $B \in M_n$.*

Proof. Part (a). If Δ is diagonal, one verifies easily that

$$\|\Delta \circ X\| = \|\overline{\Delta} \circ X\|$$

for every $X \in M_n$. The general case is reduced to this: If $A = U^* \Delta U$, where U is unitary and Δ is diagonal, then

$$\begin{aligned} \|A \circ X\| = \|(U^* \Delta U) \circ X\| \ &= \ \|\Delta \circ (UXU^*)\| \ = \ \|\Delta^* \circ (UXU^*)\| \\ = \|(U^* \Delta^* U) \circ X\| \ &= \ \|A^* \circ X\|. \end{aligned}$$

Part (b). In view of part (a) we have

$$\|B \circ X\| = \|B \circ X^*\| = \|(B^* \circ X)^*\| = \|B^* \circ X\|.$$

In view of Lemma 6.4(a), if the map f is changed from $f(X) = \gamma(X)X$ to $f(X) = \gamma(X)X^*$ or vice versa, for X in any set of normal matrices, the property (5.1) will not be affected. Therefore, and taking into account Lemma 6.3, the proof of Theorem 5.1 will be completed once we verify the following statement:

Proposition 6.5. *Let $f : M_n \to M_n$ be a map with the following properties:*

(a) *$f(X) = \gamma(X)X$ for every normal X, where $\gamma(X) \in \mathsf{T}$ may depend on X.*

(b) *$f(X) = X$ for every $X \in \mathcal{T}_n$;*

(c) *$\|A \circ B\| = \|f(A) \circ f(B)\|$ for all $A, B \in M_n$.*

Then

$$f(X) = \gamma(X)X, \quad \gamma(X) \in \mathsf{T}, \quad \forall\ X \in M_n. \tag{6.3}$$

In turn, for the proof of Proposition 6.5 a lemma will be convenient.

Lemma 6.6. *Let $A = [a_{ij}]_{i,j=1}^n$ and $B = [b_{ij}]_{i,j=1}^n$ be two $n \times n$ matrices with the property that*

$$\|A \circ X\| = \|B \circ X\| \tag{6.4}$$

for every normal matrix $X \in M_n$. Then there exists $\gamma \in \mathsf{T}$ such that either

$$a_{ii} = \gamma b_{ii}, \quad i = 1, 2, \ldots, n, \tag{6.5}$$

or

$$a_{ii} = \gamma \overline{b_{ii}}, \quad i = 1, 2, \ldots, n. \tag{6.6}$$

Proof. It will suffice to prove that

$$|a_{ii}| = |b_{ii}|, \quad i = 1, 2, \ldots, n, \tag{6.7}$$

and

$$|a_{ii} + a_{jj}| = |b_{ii} + b_{jj}|, \quad i \neq j. \tag{6.8}$$

The result then follows by Lemma 3.2.

Using (6.4) with $X = \operatorname{diag}(d_1, d_2, \ldots, d_n)$, where $d_1, d_2, \ldots, d_n$ are independent real variables, we obtain

$$\sum_{i,j=1}^{n} (d_i + d_j)^2 |a_{ij}|^2 = \sum_{i,j=1}^{n} (d_i + d_j)^2 |b_{ij}|^2. \tag{6.9}$$

Equating coefficients of $d_i d_j$ for a fixed pair of indices $i \neq j$ in (6.9), we have

$$|a_{ij}|^2 + |a_{ji}|^2 = |b_{ij}|^2 + |b_{ji}|^2, \quad i \neq j. \tag{6.10}$$

Equating coefficients of d_i^2 for a fixed i in (6.9) yields

$$4|a_{ii}|^2 + \sum_{j\neq i}(|a_{ij}|^2 + |a_{ji}|^2) = 4|b_{ii}|^2 + \sum_{j\neq i}(|b_{ij}|^2 + |b_{ji}|^2), \quad i = 1, 2, \ldots, n,$$

and taking advantage of (6.10), the equalities (6.7) follow. Now use (6.4) with a normal $X = E_{ij} + zE_{ji}$, $|z| = 1$, for a fixed pair $i \neq j$. To simplify the notation, let $i = 1$, $j = 2$. A computation shows that

$$\begin{aligned}\|A \circ X\| = {} & \left(\sum_{k=3}^{n}(|a_{k,1}|^2 + |a_{k,2}|^2 + |a_{1,k}|^2 + |a_{2,k}|^2)\right) \\ & + \left\| \begin{bmatrix} a_{21} + za_{12} & a_{11} + a_{22} \\ z(a_{11} + a_{22}) & za_{12} + a_{21} \end{bmatrix} \right\|.\end{aligned}$$

Equating with a similar expression for $\|B \circ X\|$, and using (6.10), we obtain

$$|a_{11} + a_{22}|^2 + |za_{12} + a_{21}|^2 = |b_{11} + b_{22}|^2 + |zb_{12} + b_{21}|^2.$$

In turn, use (6.7) and (6.10) to obtain

$$\mathrm{Re}\,(a_{11}\overline{a_{22}} + za_{12}\overline{a_{21}}) = \mathrm{Re}\,(b_{11}\overline{b_{22}} + zb_{12}\overline{b_{21}}),$$

or

$$\mathrm{Re}\,(a_{11}\overline{a_{22}} - b_{11}\overline{b_{22}}) = \mathrm{Re}\,\left(z(b_{12}\overline{b_{21}} - a_{12}\overline{a_{21}})\right).$$

Since this equality holds for every $z \in \mathsf{T}$, we must have

$$\mathrm{Re}\,(a_{11}\overline{a_{22}} - b_{11}\overline{b_{22}}) = 0.$$

But then $|a_{11} + a_{22}|^2 = |b_{11} + b_{22}|^2$. Analogously, $|a_{ii} + a_{jj}|^2 = |b_{ii} + b_{jj}|^2$ for any pair of distinct indices i and j. Thus, (6.8) holds, and the proof is complete. □

Lemma 6.7. *Suppose $A, B \in M_n$ are such that A is not normal. If $\|A \circ X\| = \|B \circ X\|$ for every rank one X and every normal X, then $A = \gamma B$ for some $\gamma \in \mathsf{T}$.*

Proof. Note that if $\|A \circ X\| = \|B \circ X\|$ for all rank one and for all normal matrices X, then for any unitary matrix V, the matrix pair $(C, D) = (VAV^*, VBV^*)$ also satisfies $\|C \circ X\| = \|D \circ X\|$ for all rank one and for all normal matrices X. In view of Lemma 6.6, the matrices A and B satisfy the hypotheses of Theorem 6.1. Thus, $A = \gamma B$ or $A = \gamma B^*$ for some $\gamma \in \mathsf{T}$. However, we show that the latter case is impossible. Indeed, assume $A = \gamma B^*$. Since A is not normal, it has an eigenvector $\mathbf{w}$ such that span $\{\mathbf{w}\}$ is not an orthogonally reducing subspace of A. Therefore, there is a unitary W such that $WAW^* = W\,(\gamma B^*)\,W^* = [\alpha_{ij}]_{i,j=1}^{n}$ is upper triangular with at least one nonzero off-diagonal entry in the first row. Consider $X = E_{1n}$. Then

$$WAW^* \circ X = (\alpha_{11} + \alpha_{nn})\,E_{1n}$$

and

$$WBW^* \circ X = \overline{\gamma}^{-1}\left(\sum_{i=1}^{n}\overline{\alpha_{1i}}E_{in} + \sum_{j=1}^{n}\overline{\alpha_{jn}}E_{1j}\right).$$

As $\|WAW^* \circ X\| = \|WBW^* \circ X\|$, we see that $\alpha_{1i} = 0$, $i \neq 1$, a contradiction. ☐

Now Proposition 6.5, and hence also Theorem 5.1, follows easily. Indeed, let $A \in M_n$ be a nonnormal matrix, and let $B = f(A)$. By the hypotheses of Proposition 6.5, Lemma 6.7 is applicable to the pair A, B. Hence $A = \gamma B$ for some $\gamma \in \mathsf{T}$, and the proof is complete. ☐

Proof of Theorem 5.2. Let $A, B \in M_n$. Assume first $A, B \notin \mathcal{N}_0$. The unitary invariance of norm then implies

$$\|f(A)\circ f(B)\| = \|\gamma(A)\gamma(B)W(A\circ B)^{\#}W^*\| = |\gamma(A)\gamma(B)|\cdot\|(A\circ B)^{\#}\| = \|(A\circ B)^{\#}\|.$$

However, each standard map is an isometry in Frobenius norm, so $\|(A \circ B)^{\#}\| = \|A \circ B\|$. We argue similarly if $A, B \in \mathcal{N}_0$.

Suppose lastly $A \in \mathcal{N}_0$ but $B \notin \mathcal{N}_0$. Then, A is normal, and we have

$$\begin{aligned}\|f(A)\circ f(B)\| &= \|\gamma(A)W(A^{\#})^*W^* \circ \gamma(B)WB^{\#}W^*\|\\ &= \|(A^{\#})^* \circ B^{\#}\| \ = \ \|(A^* \circ B)^{\#}\| \ = \ \|A^* \circ B\|\\ &= \|A \circ B\|,\end{aligned}$$

by (a) of Lemma 6.4. ☐

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