A note on the unitary part of a contraction

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Abstract

Short and independent proofs are given to two recent results of Gau and Wu on the unitary part of a contraction.

Keywords: Contraction, unitary, subspace, orthonormal basis, linearly independent set.

AMS Subject Classification: 15A04, 15A42, 15A45.

1 Introduction

Let M_n be the set of $n \times n$ complex matrices. Suppose $A \in M_n$ is a *contraction*, i.e., $\|Ax\| \le \|x\|$ for all $x \in \mathbf{C}^n$. By the Schur triangularization lemma, there is a unitary U such that $U^*AU = (a_{ij})$ is in upper triangular form with $|a_{11}| \le \cdots \le |a_{nn}| \le 1$. Because every column of U^*AU has norm at most one, if $|a_{11}| = \cdots = |a_{mm}| < 1 = |a_{m+1,m+1}| = \cdots = |a_{nn}|$, then $U^*AU = A_1 \oplus \operatorname{diag}(a_{m+1,m+1}, \ldots, a_{nn})$, where A_1 has spectral radius strictly less than 1, and $A_2 = \operatorname{diag}(a_{m+1,m+1}, \ldots, a_{nn})$ is unitary. The matrix A_2 is the restriction (compression) of A on the subspace S spanned by the last $n-m$ columns of U, and is called the *unitary part* of A. Suppose $j(A)$ is the smallest nonnegative integer such that

$$H_j(A) = \ker(I - A^{j*}A^j)$$

equals S, and $k(A)$ is the smallest nonnegative integer such that $H_k(A) \cap H_k(A^*)$ equals S. It was shown by Gau and Wu [2] that $j(A) \le n$ and $k(A) \le \lceil n/2 \rceil$. They also characterized those $A \in M_n$ satisfying $j(A) = n$ (respectively, $k(A) = \lceil n/2 \rceil$). Their proofs utilized results in one of their earlier papers [1], and they related the study to the concept of norm-one index for a contraction. In this note, we give short independent proofs of their results.

The author would like to thank Professors Hwa-Long Gau and Pei Yuan Wu for some helpful comments on the first draft of this note.

2 Results and proofs

Denote by $r(X)$ the spectral radius of $X \in M_n$. Following [1], we let $\mathcal{S}_n$ be the set of contractions $X \in M_n$ such that $I_n - X^*X$ has rank one and $r(X) < 1$. By the discussion in the introduction, we can always assume that a contraction $A \in M_n$ has a decomposition $A_1 \oplus A_2$ so that $A_1 \in M_m$ satisfies $r(A_1) < 1$, and $A_2 \in M_{n-m}$ is unitary.

[1]Research supported by NSF and the William and Mary Plumeri Award. Li is an honorary professor of the University of Hong Kong, and an honorary professor of the Taiyuan University of Technology.

Theorem 1 *Suppose $A \in M_n$ is a contraction with a decomposition $A_1 \oplus A_2$, where $A_1 \in M_m$ with $r(A_1) < 1$, and $A_2 \in M_{n-m}$ is unitary.*

(a) *If $j \in \{1, \ldots, m-1\}$, then $H_{j+1}(A_1) \subseteq H_j(A_1)$. The inclusion is strict if $\dim H_j(A_1) > 0$.*

(b) *We have $j(A_1) \le m$. The equality $j(A_1) = m$ holds if and only if $A_1 \in \mathcal{S}_m$.*

Consequently, $j(A) = j(A_1) \le n$. The equality $j(A) = n$ holds if and only if $A \in \mathcal{S}_n$.

Proof. (a) Note that for any $r \in \{1, \ldots, m\}$, a unit vector $x \in \mathbf{C}^m$ lies in $H_r(A_1)$ if and only if $\|A_1^r x\| = 1$, equivalently, $\|A_1 x\| = \cdots = \|A_1^r x\| = 1$. So, $H_{j+1}(A_1) \subseteq H_j(A_1)$ for any $j \in \{1, \ldots, m-1\}$.

Suppose $\dim H_j(A_1) = \ell > 0$. Assume the contrary that $\dim H_{j+1}(A_1) = \ell$. Let $\{x_1, \ldots, x_\ell\}$ be an orthonormal basis for $H_{j+1}(A_1)$. For $i = 1, \ldots, \ell$, we have $1 = \|A_1^{j+1} x_i\| = \|A_1^j (A_1 x_i)\|$ so that $\{A_1 x_1, \ldots, A_1 x_\ell\} \subseteq H_j(A_1) = \operatorname{span}\{x_1, \ldots, x_\ell\}$. Thus, if $U \in M_m$ is unitary such that $x_1, \ldots, x_\ell$ are the first ℓ columns of U, then $U^* A_1 U$ has the form $\begin{pmatrix} A_{11} & A_{12} \\ 0 & A_{22} \end{pmatrix}$ with $A_{11} \in M_\ell$. Since A_1 is a contraction and each column of A_{11} is a unit vector, we see that $A_{12} = 0$ and A_{11} is unitary, which contradicts the fact that $r(A_1) < 1$.

(b) Since A_1 is a contraction with $r(A_1) < 1$, by (a) we have

$$\dim H_m(A_1) < \cdots < \dim H_1(A_1) < \dim H_0(A_1) = m,$$

if none of $\dim_j(A_1)$ is 0 for $j = 1, \ldots, m-1$. Thus, $j(A_1) \le m$. Furthermore, if $j(A_1) = m$, then $\dim H_j(A_1) = m - j$ for $j = 1, \ldots, m$. Hence, $\dim H_1(A_1) = m - 1$, i.e., $I_m - A_1^* A_1$ has rank 1. Hence, $A_1 \in \mathcal{S}_m$.

Conversely, suppose $A_1 \in \mathcal{S}_m$. Then for $V = H_1(A_1)$, we have $\dim[A_1(V) \cap V] \ge m - 2$. Inductively, we have $\dim[A_1^j(V) \cap V] \ge m - j - 1$. Thus, $j = m - 1$ is the smallest integer such that $\dim[A_1^j(V) \cap V] = 0$ so that $H_{j+1}(A_1) = \{0\}$. Hence, $j(A_1) = m$.

Note that $j(A) = j(A_1)$. The last assertion follows readily from (a) and (b). $\square$

Theorem 2 *Suppose $A \in M_n$ is a contraction with a decomposition $A_1 \oplus A_2$, where $A_1 \in M_m$ with $r(A_1) < 1$, and $A_2 \in M_{n-m}$ is unitary.*

(a) *If k is a nonnegative integer such that $k \le m/2$, then*

$$\dim [H_{k+1}(A_1) \cap H_{k+1}(A_1^*)] \le \max\{0, \dim [H_k(A_1) \cap H_k(A_1^*)] - 2\}.$$

(b) *We have $k(A_1) \le \lceil m/2 \rceil$. The equality $k(A_1) = \lceil m/2 \rceil$ holds if and only if*

(i) $A_1 \in \mathcal{S}_m$, *or* (ii) *m is even and $\|A_1^{m-2}\| = 1 > \|A_1^{m-1}\|$.*

Consequently, $k(A) = k(A_1) \le \lceil n/2 \rceil$. The equality $k(A) = \lceil n/2 \rceil$ holds if and only if one of the following holds.

(1) *$A \in \mathcal{S}_n$.*
(2) *n is even and A is unitarily similar to $[e^{it}] \oplus A_1$ with $t \in \mathbf{R}$ and $A_1 \in \mathcal{S}_{n-1}$.*
(3) *n is even, $\|A^{n-2}\| = 1 > \|A^{n-1}\|$.*

Proof. (a) Let $V_k = H_k(A_1) \cap H_k(A_1^*)$. Then a unit vector $x \in \mathbf{C}^m$ lies in V_k if and only if $\|B^k x\| = 1$ for $B \in \{A_1, A_1^*\}$, equivalently, $\|B^r x\| = 1$ for $r = 1, \ldots, k$. Thus, $V_{k+1} \subseteq V_k$.

Suppose $\{x_1, \ldots, x_p\}$ is an orthonormal basis for V_k so that $\{x_1, \ldots, x_\ell\}$ is a basis for V_{k+1}. We **claim** that $\ell \le \max\{0, p-2\}$.

Suppose the claim is not true, and $\ell \in \{p, p-1\}$ with $\ell > 0$. Since A_1 is a contraction and $1 = \|A_1^r x_j\|$ for $r = 1, \ldots, k$, we see that $A_1^{*r} A_1^r x_j = x_j$ for $j = 1, \ldots, p$. For $i \in \{1, \ldots, \ell\}$, we have $\|A_1^k(A_1 x_i)\| = \|A_1^{k+1} x_i\| = 1$ and $\|A_1^{*k}(A_1 x_i)\| = \|A_1^{*(k-1)}(A_1^* A_1 x_i)\| = \|A_1^{*(k-1)} x_i\| = 1$. Thus, $\{A_1 x_1, \ldots, A_1 x_\ell\} \subseteq V_k = \operatorname{span}\{x_1, \ldots, x_p\}$.

Let $U \in M_m$ be such that $x_1, \ldots, x_p$ are the first p columns of U, and $\tilde{A}_1 = U^* A_1 U$ has the form $\tilde{A}_1 = \begin{pmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{pmatrix}$ with $A_{11} \in M_p$. Then each of the first p columns (respectively, rows) of $\tilde{A}_1$ has unit length.

If $\ell = p$, then A_{12} and A_{21} are zero matrices, and the columns of A_{11} have unit length. Since A_1 is a contraction, A_{11} is unitary, which contradicts the fact that $r(A_1) < 1$.

If $\ell = p-1$, then only the last column of A_{21} and the last row of A_{12} can be nonzero. Moreover, $\tilde{A}_1^* \tilde{A}_1 = I_p \oplus C$ for some $C \in M_{m-p}$ so that $A_{11}^* A_{11} = I_{p-1} \oplus [\mu]$ for some $\mu \in [0,1]$. Similarly, we have $\tilde{A}_1 \tilde{A}_1^* = I_{p-1} \oplus \tilde{C}$ for some $\tilde{C} \in M_{m-p}$ so that $A_{11} A_{11}^* = I_{p-1} \oplus [\mu]$. Clearly, we have $\mu < 1$. Otherwise, $\|B^{k+1} x_p\| = 1$ for $B \in \{A_1, A_1^*\}$ so that $x_p \in V_{k+1}$. Consequently, $A_{11} = W \oplus [\gamma]$, where $|\gamma|^2 = \mu$ and $W \in M_{p-1}$ is unitary, which contradicts the fact that $r(A_1) < 1$.

By the above discussion, we see that $\ell \le \max\{0, p-2\}$.

(b) By (a), $m \ge \dim V_1 + 2 \ge \dim V_2 + 4 \ge \cdots \ge \dim V_j + 2j \ge \cdots$. Thus, if $m = 2k$ or $m = 2k-1$, then $k(A_1) \le k$.

If $A_1 \in \mathcal{S}_m$, then $\|A^{m-1}\| = 1$ by Theorem 1. Thus, there is a unit vector $x \in \mathbf{C}^m$ such that $x, A_1 x, \ldots, A_1^{m-1} x$ are unit vectors. If $m = 2k-1$, then $A_1^{k-1}(A_1^{k-1} x) = A_1^{2k-2} x$ and $(A_1^*)^{k-1} A_1^{k-1} x = x$ are unit vectors. So, $A_1^{k-1} x \in V_{k-1}$, and hence $V_{k-1} \neq \{0\}$. Thus, k is the smallest integer satisfying $V_k = \{0\}$. Similarly, if $m = 2k$ and $\|A_1^{m-2}\| = 1$, then $A_1^{k-1} x \in V_{k-1}$ so that k is the smallest integer satisfying $V_k = \{0\}$.

Conversely, if $m = 2k$ or $m = 2k-1$, and V_{k-1} is nonzero, then there is a unit vector $x \in V_{k-1}$, i.e., $A_1^{k-1} x$ and $(A_1^*)^{k-1} x$ are unit vectors. Since A_1^{k-1} is a contraction, we see that $(A_1^{k-1})^* A_1^{k-1} x = x$. Thus $(A_1^*)^{2k-2}(A_1^{k-1} x) = (A_1^*)^{k-1} x$ is a unit vector. If $m = 2k-1$, then $\|A_1^{m-1}\| = 1$; by Theorem 1, $A \in \mathcal{S}_m$. If $m = 2k$, then $\|A_1^{m-2}\| = 1$; moreover, either $\|A_1^{m-1}\| = 1$ so that $A_1 \in \mathcal{S}_m$, or $\|A_1^{m-1}\| < 1$.

Note that $k(A) = k(A_1)$. The last assertion follows readily from (a) and (b). $\square$

References

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[2] H.-L. Gau, P.Y. Wu, Unitary part of a contraction, J. Math. Anal. Appl. 366 (2010), 700-705.