Multiplicative Preservers of C-Numerical Ranges and Radii

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Abstract

Multiplicative preservers of C-numerical ranges and radii on certain groups and semigroups of complex $n \times n$ matrices are characterized. The general and special linear groups are considered, as well as the semigroups of matrices having ranks not exceeding k, with k fixed in advance. For a fixed C, it turns out that typically the multiplicative preservers of the C-numerical range (or radius) have the form $A \mapsto f(\det A)UAU^*$ or, for certain matrices C, the form $A \mapsto f(\det A)U\overline{A}U^*$, for some unitary U and multiplicative map f from the group of nonzero complex numbers to the unit circle.

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1 Introduction

Let M_n be the algebra of complex $n \times n$ matrices. Given a nonzero $C \in M_n$, the C-numerical range and the C-numerical radius of $A \in M_n$ are defined by

$$W_C(A) = \{\mathrm{tr}\,(CUAU^*) : U \text{ unitary }\}$$

and

$$w_C(A) = \max\{|z| : z \in W_C(A)\}.$$

The concepts of C-numerical range and C-numerical radius were introduced by Goldberg and Strauss; see [5] and [6]. When C is a rank one Hermitian orthogonal projection, $W_C(A)$ and $w_C(A)$ reduce to the classical numerical range $W(A) = \{x^*Ax : x \in \mathbb{C}^n, x^*x = 1\}$ and the classical numerical radius $w(A) = \max\{|z| : z \in W(A)\}$, which are useful concepts in studying matrices and operators; see, for example, [11]. The C-numerical range and the C-numerical radius are also very useful in studying matrices and operators, and have attracted

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the attention of many researchers; see [12] and its references. For example, it was proved in [16] that C-numerical radii can be viewed as the building blocks of unitary similarity invariant norms $\|\cdot\|$ on M_n, i.e., norms on M_n satisfying $\|A\| = \|U^*AU\|$ for any $A \in M_n$ and unitary U, in the sense that *for any unitary similarity invariant norm* $\|\cdot\|$ *on* M_n *there is a compact subset* $S \subseteq M_n$ *such that*

$$\|A\| = \max\{w_C(A) : C \in S\}.$$

An interesting topic in the study of C-numerical range and C-numerical radius is characterizing linear maps $\phi : M_n \to M_n$ such that $F(\phi(A)) = F(A)$ for all $A \in M_n$, where $F(A) = w_C(A)$ or $W_C(A)$; see [12, 14, 15, 20]. Such maps are called linear preservers of $F(A)$. If $w_C(A)$ is a norm, then linear preservers of $w_C(A)$ are just linear isometries of $w_C(A)$. Furthermore, in most of the cases linear preservers of $W_C(A)$ have the form

$$A \mapsto U^*AU, \qquad U \text{ is a fixed unitary matrix,}$$

which is also multiplicative, i.e., $\phi(AB) = \phi(A)\phi(B)$ for all $A, B \in M_n$.

Recently, there has been considerable interest also in studying multiplicative maps on groups and semigroups of matrices that leave invariant some special functions, sets, and relations, see, for example, [10, 2, 7, 8, 9]. The approach undertaken in [9] is based on the classical results of Borel - Tits on automorphisms of linear groups.

In this paper, we characterize multiplicative preservers of C-numerical ranges and radii on the following semigroups of M_n:

SL_n: the group of matrices in M_n with determinant 1,
GL_n: the group of invertible matrices in M_n,
$M_n^{(k)}$: the semigroup of matrices in M_n with rank at most k.

In more detail, for a fixed $C \in M_n$, we characterize those multiplicative maps $\phi : \mathbf{H} \longrightarrow M_n$, where $\mathbf{H}$ is one of SL_n, GL_n, or $M_n^{(k)}$, that have the property

$$w_C(\phi(A)) = w_C(A) \qquad \text{for every } A \in \mathbf{H}, \tag{1.1}$$

or the property

$$W_C(\phi(A)) = W_C(A) \qquad \text{for every } A \in \mathbf{H}. \tag{1.2}$$

Multiplicative maps with the property (1.1), resp. (1.2), will be called *multiplicative preservers* of $w_C(A)$, resp. of $W_C(A)$.

The following notation will be used.

$\mathbb{C}$ the complex field.
$\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ the multiplicative group of nonzero complex numbers.
$\mathbf{T}$ the unit circle in $\mathbb{C}$.
$\sigma : \mathbb{C} \longrightarrow \mathbb{C}$ a complex field embedding.
$\{E_{1,1}, E_{1,2}, \ldots, E_{n,n}\}$ the standard basis for M_n.
I_m (or I with m understood) the $m \times m$ identity matrix.
0_m (or 0 with m understood) the $m \times m$ zero matrix.

$w(A)$ the numerical radius of $A \in M_n$.
$\operatorname{tr} A$ the trace of $A \in M_n$.
A^t the transpose of A.
$\tau(A) = (A^{-1})^t$, for an invertible matrix A.
$\overline{A}$ the entrywise complex conjugate of a matrix A.
$A^* = (\overline{A})^t$.
$s_1(X) \geq s_2(X) \geq \cdots \geq s_n(X)$ the singular values of a matrix $X \in M_n$.
$\operatorname{diag}(a_1, \ldots, a_n)$ diagonal matrix with the diagonal entries $a_1, \ldots, a_n$ (in that order).

If $C = \mu I$, $\mu \neq 0$, then $W_C(A) = \{\mu \operatorname{tr}(A)\}$. So, the problem of describing the multiplicative preservers of $w_C(A)$ or of $W_C(A)$ reduces to the multiplicative preservers of $|\operatorname{tr} A|$ or of $\operatorname{tr} A$ which was treated in [9] for the cases of SL_n and of GL_n and followed readily from Proposition 3.7 in [2] for the case of $M_n^{(k)}$. For $\operatorname{tr} A$, the multiplicative preservers have the form:

(i) $A \mapsto SAS^{-1}$ for some $S \in SL_n$.

For $|\operatorname{tr} A|$, the multiplicative preservers have form (i) or the following forms:

(ii) $A \mapsto S\overline{A}S^{-1}$,
(iii) $A \mapsto f(\det(A))SAS^{-1}$ or $A \mapsto f(\det(A))S\overline{A}S^{-1}$ for preservers on GL_n,

for some $S \in SL_n$ and multiplicative map $f : \mathbb{C}^* \to \mathbf{T}$.

Thus, in the sequel we *always implicitly assume that C is not a scalar matrix.* Also, we will always assume that $n \geq 2$ to avoid trivial considerations.

2 Preliminaries

The following easily verified property of the C-numerical ranges will be used:

$$\overline{W_{\overline{C}}(A)} = W_C(\overline{A}), \qquad A \in M_n, \tag{2.1}$$

and therefore

$$w_{\overline{C}}(A) = w_C(\overline{A}), \qquad A \in M_n. \tag{2.2}$$

We need also Proposition 2.1 and Lemma 2.3 below from [9], and a well-known (and easily proved) Lemma 2.2.

Proposition 2.1 *Let ϕ be a multiplicative map on SL_n or GL_n. Then either $\phi(SL_n)$ is a singleton or there exist a field embedding $\sigma : \mathbb{C} \to \mathbb{C}$, a matrix $S \in SL_n$, and a multiplicative map $f : \mathbb{C}^* \to \mathbb{C}^*$ such that ϕ has the form*

$$A \mapsto f(\det(A))S\sigma(A)S^{-1} \quad \text{or} \quad A \mapsto f(\det(A))S\tau(\sigma(A))S^{-1}.$$

Lemma 2.2 *Suppose k is a positive integer, and $f : \mathbb{C}^* \to \mathbb{C}^*$ is a group homomorphism such that $f(\mu)^k = 1$ for all $\mu \in \mathbb{C}^*$. Then $f(\mu) = 1$ for all $\mu \in \mathbb{C}^*$.*

Lemma 2.3 *Let $S \in SL_n$. If $SE_{ij}S^{-1}$ has singular values $1, 0, \ldots, 0$ for all $i, j \in \{1, \ldots, n\}$ with $i \neq j$, then S is unitary.*

The following characterization of the continuous complex field embeddings is useful; see, e.g., [22].

Lemma 2.4 *The following statements for a complex field embedding $\sigma : \mathbb{C} \to \mathbb{C}$ are equivalent:*

(a) *either $\sigma(z) = z$ for every $z \in \mathbb{C}$ or $\sigma(z) = \overline{z}$ for every $z \in \mathbb{C}$.*

(b) *$|\sigma(z)| = |z|$ for every $z \in \mathbb{C}$.*

(c) *$\sigma(z) > 0$ for every positive z.*

(d) *σ is continuous.*

3 C-numerical radius preservers on SL_n and GL_n

We need the following lemma to characterize multiplicative preservers of C-numerical radius.

Lemma 3.1 *Let $C \in M_n$. Then $w_C(A) = w_C(\overline{A})$ for all $A \in M_n$ if and only if C is unitarily similar to $\mu\overline{C}$ for some $\mu \in \mathbf{T}$.*

Proof. Since

$$w_A(C) = w_C(A) = w_C(\overline{A}) = w_{\overline{C}}(A) = w_A(\overline{C}) \qquad \text{for all } A \in M_n,$$

by Theorem 2.1 in [16], we see that $\overline{C}$ is in the convex hull of the set

$$\{\mu U^*CU : \mu \in \mathbf{T},\ U^*U = I_n\}. \tag{3.1}$$

Thus, $\overline{C}$ is a convex combination of a finite subset of the set (3.1). Since μU^*CU (for $|\mu| = 1$ and unitary U) and $\overline{C}$ have the same Frobenius norm, and the Frobenius norm is strictly convex, we must have that $\overline{C} = \mu U^*CU$ for some $\mu \in \mathbf{T}$ and unitary U. ∎

Theorem 3.2 *Let $\mathbf{H} = SL_n$ or GL_n, and let $C \in M_n$ be a fixed non-scalar matrix. A multiplicative map $\phi : \mathbf{H} \to M_n$ satisfies $w_C(\phi(A)) = w_C(A)$ for all $A \in \mathbf{H}$ if and only if there is a unitary $U \in SL_n$ and a multiplicative map $f : \mathbb{C}^* \to \mathbf{T}$ such that one of the following conditions holds true:*

(a) *ϕ has the form $A \mapsto f(\det(A))UAU^*$.*

(b) *There exists $\mu \in \mathbf{T}$ such that C and $\mu\overline{C}$ are unitarily similar, and ϕ has the form $A \mapsto f(\det(A))U\overline{A}U^*$.*

Note that $f(\det(A)) = 1$ for every $A \in SL_n$.

Proof. The "if" part can be verified with the help of Lemma 3.1. We focus on the converse.

Suppose $\mathrm{tr}\,(C) \neq 0$. Then $w_C(A)$ is a unitary similarity invariant norm; see [6], also [16]. By [9, Theorem 3.8], ϕ has the asserted form, and the form of ϕ in (b) holds if and only if $w_C(A) = w_C(\overline{A})$ for all $A \in \mathbf{H}$. Since every $A \in GL_n$ is a scalar multiple of $B \in SL_n$, and GL_n is dense in M_n, by continuity of the norm function w_C we see that if $w_C(A) = w_C(\overline{A})$ for all $A \in \mathbf{H}$, then $w_C(A) = w_C(\overline{A})$ for all $A \in M_n$. By Lemma 3.1, we see that $\overline{C}$ is unitarily similar to μC for some $\mu \in \mathbf{T}$.

Now, suppose that $\mathrm{tr}\,(C) = 0$. Since C is not a scalar matrix,

$$\gamma = w_C(E_{1,2}) = w_C(E_{i,j}) > 0, \quad i \neq j.$$

For any A unitarily similar to $I + \nu E_{n1}$, we have:

$$\begin{aligned} w_C(A) &= \max\{|\mathrm{tr}\,(CVAV^*)| : V \text{ is unitary}\} \\ &= \max\{|\mathrm{tr}\,(V^*CV(I + \nu E_{n1})| : V \text{ is unitary}\} \\ &= |\nu|\gamma. \end{aligned} \tag{3.2}$$

Suppose $\mathbf{H} = SL_n$. Clearly, ϕ is non-trivial on $\mathbf{H}$, and hence by Proposition 2.1 ϕ has the standard form

$$A \mapsto S\sigma(A)S^{-1} \quad \text{or} \quad A \mapsto S\tau(\sigma(A))S^{-1}$$

for some $S \in SL_n$. Suppose S is not unitary. By Lemma 2.3, there is $E_{i,j}$ with $i \neq j$ such that $SE_{i,j}S^{-1}$ has trace zero and singular values $r, 0, \ldots, 0$ with $r \neq 1$. But then for $A = I + E_{i,j}$, we have by (3.2):

$$w_C(A) = 1 + \gamma \neq 1 + r\gamma = w_C(\phi(A)),$$

which is a contradiction. Hence, S is unitary. Furthermore, for any $z \in \mathbb{C}$, if $A_z = I + zE_{1,n}$ then again by (3.2)

$$\gamma|z| = w_C(A_z) = w_C(\phi(A)) = \gamma|\sigma(z)|.$$

So, by Lemma 2.4 σ has the form $z \mapsto z$ or $z \mapsto \bar{z}$.

For $n > 2$ we show that ϕ cannot have the form $A \mapsto S\tau(\sigma(A))S^{-1}$. (For $n = 2$ the form $A \mapsto S\tau(\sigma(A))S^{-1}$ is essentially the same as the form $A \mapsto SAS^{-1}$.) Note that there is a unitary matrix V such that $V^*CV = (\gamma_{ij})$ with $|\gamma_{11}| = w(C)$. If

$$A = \mathrm{diag}\,(m^2, 1/m, 1/m, 1, \ldots, 1)$$

for sufficiently large m, then

$$w_C(A) \geq |\mathrm{tr}\,(V^*CVA)| \geq m^2|\gamma_{11}| - \sum_{j=2}^{n} |\gamma_{jj}| > w(C)(m^2 - n). \tag{3.3}$$

On the other hand, if a unitary U having columns $u_1, \ldots, u_n$ is such that

$$w_C(S\tau(A)S^{-1}) = |\mathrm{tr}\,(U^*CU\tau(A))|,$$

then

$$\begin{aligned} w_C(S\tau(A)S^{-1}) &\le \frac{1}{m^2}|u_1^*Cu_1| + m|u_2^*Cu_2| + m|u_3^*Cu_3| + |u_4^*Cu_4| + \cdots + |u_n^*Cu_n| \\ &\le w(C)(\frac{1}{m^2} + 2m + n - 3), \end{aligned}$$

which is smaller than the right hand side of (3.3). So, ϕ has the asserted form

$$A \mapsto UAU^* \quad \text{or} \quad A \mapsto U\overline{A}U^* \tag{3.4}$$

for some unitary matrix U. If the latter holds, then $w_C(A) = w_C(\overline{A})$ for all $A \in SL_n$. By continuity and homogeneity, $w_C(A) = w_C(\overline{A}) = w_{\overline{C}}(A)$ for all $A \in M_n$. By Lemma 3.1, we see that C and $\mu\overline{C}$ are unitarily similar for some $\mu \in \mathbf{T}$.

Now, suppose $\mathbf{H} = GL_n$. Then by Proposition 2.1 ϕ has the form

$$A \mapsto f(\det(A))UAU^* \quad \text{or} \quad A \mapsto f(\det(A))U\overline{A}U^*, \tag{3.5}$$

where the latter case holds when $w_C(A) = w_C(\overline{A})$ for all $A \in M_n$, and where $f : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ is a multiplicative map. Then for any $z \in \mathbb{C}^* \setminus \{-1\}$ and $A_z = I + zE_{1,1}$,

$$\begin{aligned} w_C(A) &= \max\{|\mathrm{tr}\,(CU^*AU)| : U \text{ is unitary}\} \\ &= \max\{|z\mathrm{tr}\,(CU^*E_{1,1}U)| : U \text{ is unitary}\} \\ &= w(zC). \end{aligned}$$

Similarly, $w_C(\phi(A)) = |f(1+z)|w(zC)$. Thus, $|f(\mu)| = 1$ for all $\mu \in \mathbb{C}^*$. ∎

4 C-numerical range preservers on SL_n and GL_n

We need some additional facts to state and prove the results on multiplicative preservers of C-numerical range. First, recall that a block matrix (X_{ij}) is in *block shift form* if all the diagonal blocks are square matrices (may be of different sizes) and $X_{ij} = 0$ whenever $j \neq i+1$. This is a generalization of the weighted shift matrix where all X_{ij} are one by one. We have the following result; see [17].

Lemma 4.1 *The following conditions are equivalent for a non-scalar matrix C:*

(a) *C is unitarily similar to a matrix in block shift form.*

(b) *$W_C(A)$ is a circular disk centered at the origin for all $A \in M_n$.*

(c) *$W_C(C^*)$ is a circular disk centered at the origin.*

Theorem 4.2 *Let $\mathbf{H} = SL_n$ or GL_n, and let $C \in M_n$ be a non-scalar matrix. A multiplicative map $\phi : \mathbf{H} \to M_n$ satisfies $W_C(\phi(A)) = W_C(A)$ for all $A \in \mathbf{H}$ if and only if there is a unitary $U \in SL_n$ and a multiplicative map $f : \mathbb{C}^* \to \mathbf{T}$ such that one of the following holds true.*

(a) *ϕ has the form $A \mapsto UAU^*$.*

(b) *C is unitarily similar to a matrix in block shift form and ϕ has the form*

$$A \mapsto f(\det(A))UAU^*.$$

(c) *C is unitarily similar to a matrix in block shift form, as well as unitarily similar to $\mu\overline{C}$ for some $\mu \in \mathbf{T}$, and ϕ has the form $A \mapsto f(\det(A))U\overline{A}U^*$.*

Proof. The "if" part can be verified readily with the help of Lemmas 3.1 and 4.1. We consider the converse.

Suppose $\mathbf{H} = SL_n$. Note that the $W_C(A)$ preservers must also be $w_C(A)$ preservers. Thus, ϕ has the form described in Theorem 3.2. Suppose ϕ has the form

$$A \mapsto U\overline{A}U^*, \qquad U \text{ unitary},$$

and C is unitarily similar to $\mu\overline{C}$ for some $\mu \in \mathbf{T}$. Now, for any $A \in SL_n$, then

$$W_C(A) = W(\phi(A)) = W_C(\overline{A}) = \mu W_{\overline{C}}(\overline{A}) = \mu\overline{W_C(A)}. \tag{4.1}$$

We claim that C is unitarily similar to a matrix in block shift form. First, we show that $\operatorname{tr} C = 0$. Note for $\xi = e^{i2\pi/n}$, $\xi I \in SL_n$ and

$$\{\xi \operatorname{tr} C\} = W_C(\xi I) = W_C(\bar{\xi} I) = \{\bar{\xi} \operatorname{tr} C\}.$$

Thus, $\xi^2 \operatorname{tr} C = \operatorname{tr} C$. If $n > 2$, then $\operatorname{tr} C = 0$. If $n = 2$, then (see [19] and [13, Theorem 1]) $W_C(A)$ is an elliptical disk centered at $(\operatorname{tr} C)(\operatorname{tr} A)/2$. If $\operatorname{tr} C \neq 0$, one can choose

$$A = \begin{pmatrix} \mu & \mu^2 - 1 \\ 1 & \mu \end{pmatrix} \in SL_2$$

such that $\mu \operatorname{tr} C \neq \bar{\mu} \operatorname{tr} C$. Thus, $W_C(A) \neq W_C(\overline{A})$, which is a contradiction.

Now, $\operatorname{tr} C = 0$. Suppose $A \in GL_n$ with eigenvalues $\alpha_1, \ldots, \alpha_n$. For any $\gamma \in \mathbb{C}$ which does not coincide with any of $-\alpha_j$, we have

$$X := \frac{A + \gamma I}{\left[\prod_j (\alpha_j + \gamma)\right]^{1/n}} \in SL_n.$$

Thus, using the property that $\operatorname{tr} C = 0$ and (4.1), we have

$$\left[\prod_j (\alpha_j + \gamma)\right]^{-1/n} W_C(A) = \left[\prod_j (\alpha_j + \gamma)\right]^{-1/n} W_C(A + \gamma I)$$

$$= W_C(X) = W_C(\overline{X}) = \left[\overline{\prod_j (\alpha_j + \gamma)}\right]^{-1/n} W_C(\overline{A}).$$

Letting $\delta = ((\alpha_1 + \gamma) \ldots (\alpha_n + \gamma))^{-1/n}$, it now follows that

$$\left(\delta/\overline{\delta}\right) W_C(A) = W_C(\overline{A}).$$

It is easy to see that $\delta/\overline{\delta}$ can be made equal to any prescribed number in $\mathbf{T}$, for a suitable choice of γ. Since $W_C(A)$ is start-shaped (see [3]), it follows that $W_C(A)$ is a circular disk centered at the origin. Now, for any $A \in M_n$, there is $\lambda \in \mathbb{C}$ such that $A + \lambda I \in GL_n$, and

$$W_C(A) = W_C(A + \lambda I)$$

is a circular disk centered at origin. By Lemma 4.1, C is unitarily similar to a matrix in block shift form.

Next, suppose $\mathbf{H} = GL_n$. Since ϕ preserves $w_C(A)$, by Theorem 3.2 ϕ has the form

$$A \mapsto f(\det(A)) UAU^* \quad \text{or} \quad A \mapsto f(\det(A)) U\overline{A}U^* \tag{4.2}$$

for some multiplicative map $f : \mathbb{C}^* \to \mathbf{T}$, and some unitary U. If ϕ has the second form in (4.2), then by restricting ϕ to SL_n, and applying the result on SL_n we already proved, we conclude that C is unitarily similar to a block shift form, as well as to $\mu\overline{C}$ for some $\mu \in \mathbf{T}$.

It remains to show that if ϕ has the form $A \mapsto f(\det(A))UAU^*$, where f is non-trivial, then C is unitarily similar to a block shift form. By Lemma 2.2, it is easy to see that the range of f is dense in $\mathbf{T}$. For every $z \in \mathbb{C}^*$ and every $A \in GL_n$, we have:

$$zW_C(A) = W_C(zA) = W_C(\phi(zA)) = f(z^n(\det A))W_C(zA) = zf(z^n(\det A))W_C(A).$$

Thus, $f(z^n(\det A))W_C(A) = W_C(A)$, and by the denseness of the range of f we conclude that

$$\nu W_C(A) = W_C(A) \qquad \text{for every } \nu \in \mathbf{T}.$$

Since $W_C(A)$ is start-shaped (see [3]), it follows that $W_C(A)$ is a circular disk centered at the origin. Now, $\{\nu \operatorname{tr} C\} = \nu W_C(I) = W_C(I) = \{\operatorname{tr} C\}$ for every $\nu \in \mathbf{T}$; so, $\operatorname{tr} C = 0$. Furthermore, for any $A \in M_n$, there is $\lambda \in \mathbb{C}$ such that $A + \lambda I \in GL_n$; then

$$W_C(A) = W_C(A + \lambda I)$$

is a circular disk centered at origin. By Lemma 4.1, C is unitarily similar to a matrix in block shift form. ∎

In connection with Theorem 4.2 the following example is instructive.

Example 4.3 We construct here a family of examples of block shift matrices A such that A is not unitarily similar to $\mu\overline{A}$, for any $\mu \in \mathbf{T}$; in particular, A not unitarily similar to any real matrix.

We start with general observations:

1. *Every diagonalizable matrix with positive eigenvalues is a product of two positive definite matrices.*

This fact is well-known; for a proof note that if $X = S^{-1}DS$, where S is invertible and D is diagonal with positive numbers on the diagonal, then $X = S^{-1}(S^{-1})^* \cdot S^*DS$ is a product of two positive definite matrices.

A *word* $w(X, Y)$, where X and Y are $n \times n$ matrices, is any matrix of the form

$$w(X, Y) = X^{\alpha_1}Y^{\beta_1}X^{\alpha_2}Y^{\beta_2}\cdots X^{\alpha_p}Y^{\beta_p},$$

where α_j, β_j are nonnegative integers. The integer $\sum_{j=1}^{p}(\alpha_j - \beta_j)$ will be called the *index* of $w(X, Y)$.

2. *If C is unitarily similar to $\mu\overline{C}$, for some $\mu \in \mathbf{T}$, then* $\mathrm{tr}\,(w(C^*, C))$ *is real for every word $w(C^*, C)$ with zero index.*

The proof is elementary: Assume $C = U(\mu\overline{C})U^*$ for some unitary U and $\mu \in \mathbf{T}$. Then

$$w(C^*, C) = Uw(\overline{\mu}C^t, \mu\overline{C})U^* = \overline{\overline{U}(w(\mu\overline{C^t}, \overline{\mu}C))\overline{U}^*} = \overline{\overline{U}(w(\mu C^*, \overline{\mu}C))\overline{U}^*},$$

which is equal to $\overline{\overline{U}(w(C^*, C))\overline{U}^*}$, assuming that the index of $w(C^*, C)$ is zero. Thus, $\mathrm{tr}\,w(C^*, C) = \overline{\mathrm{tr}\,w(C^*, C)}$, and 2. follows.

To construct the matrix A as required, we let A_1, A_2, A_3 be 2×2 positive definite matrices such that

$$A_1A_2 = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix},$$

where a and c are distinct positive numbers and $b \neq 0$ is real, and the off-diagonal entries of A_3 are non-real (the existence of A_1 and A_2 with the required properties follows from Fact 1.). Then

$$\mathrm{tr}\,(A_1A_2A_3) \notin \mathbb{R}. \tag{4.3}$$

Next, let $A_{1,2}$, $A_{2,3}$ and $A_{3,4}$ be such that

$$A_{2,3}A_{2,3}^* = A_1, \quad A_{1,2}^*A_{1,2} = A_2, \quad A_{2,3}A_{3,4}^*A_{3,4}A_{2,3}^* = A_3,$$

and finally

$$A = \begin{pmatrix} 0 & A_{1,2} & 0 & 0 \\ 0 & 0 & A_{2,3} & 0 \\ 0 & 0 & 0 & A_{3,4} \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

A computation shows that

$$\mathrm{tr}\,(A(A^*)^2A^3(A^*)^2) = \mathrm{tr}\,(A_1A_2A_3),$$

and by Fact 2. and (4.3), A cannot be unitarily similar to $\mu\overline{A}$ for any $\mu \in \mathbf{T}$.

5 Results on $M_n^{(k)}$

We start with a preliminary result. Matrices $X_1, \ldots, X_n \in M_n$ are said to be *mutually orthogonal rank one idempotents* if $X_i^2 = X_i$ and $X_iX_j = 0$ for any $i, j \in \{1, \ldots, n\}$ with $i \neq j$. We have the following fact from [2, Propositions 2.2 and 2.3].

Proposition 5.1 *Let $\phi : M_n^{(k)} \to M_n$ be a multiplicative map. Then there exist mutually orthogonal rank one idempotents $X_1, \ldots, X_n$ such that $\phi(X_i) \neq \phi(0)$ for $i = 1, \ldots, n$ if and only if there exist $S \in SL_n$ and a field embedding $\sigma : \mathbb{C} \to \mathbb{C}$ such that ϕ has the form*

$$(a_{ij}) \mapsto S(\sigma(a_{ij}))S^{-1}.$$

Our main result on multiplicative preservers of the C-numerical ranges and radii on $M_n^{(k)}$ reads as follows.

Theorem 5.2 *Let $C \in M_n$ be a non-scalar matrix, and let $F_C(A) = w_C(A)$ or $W_C(A)$. A multiplicative map $\phi : M_n^{(k)} \to M_n$ satisfies $F_C(\phi(A)) = F_C(A)$ for all $A \in M_n^{(k)}$ if and only if there is a unitary $U \in SL_n$ such that one of the following conditions holds true:*

(a) *ϕ has the form $A \mapsto UAU^*$.*

(b) *$F_C(A) = F_C(\overline{A})$ for all $A \in M_n^{(k)}$, and ϕ has the form $A \mapsto U\overline{A}U^*$.*

Proof. The "if" part can be verified readily. We focus on the converse.

For $i = 1, \ldots, n$, we have $w_C(E_{i,i}) = w(C) \neq 0 = w_C(0)$. Thus, $\phi(E_{i,i}) \neq \phi(0)$ for $i = 1, \ldots, n$. By Proposition 5.1 ϕ has the form

$$\phi(A) = S\sigma(A)S^{-1}, \qquad A \in M_n^{(k)},$$

where $S \in SL_n$. Suppose S is not unitary. By Lemma 2.3, there is $E_{i,j}$ with $i \neq j$ such that $SE_{i,j}S^{-1}$ is unitarily similar to $rE_{1,2}$ with some positive real number $r \neq 1$. Since C is not a scalar matrix, we have

$$0 < w_C(E_{i,j}) \neq rw_C(E_{1,2}) = w_C(\phi(E_{i,j})),$$

which is a contradiction. Hence, S is unitary. Furthermore, for any $z \in \mathbb{C}$,

$$|z|w_C(E_{1,2}) = w_C(zE_{1,2}) = w_C(\sigma(z)E_{1,2}) = |\sigma(z)|w_C(E_{1,2}).$$

So, by Lemma 2.4 σ has the form $z \mapsto z$ or $z \mapsto \bar{z}$. The result follows. ∎

Theorem 5.2 is not entirely satisfactory as we do not have a complete characterization of the sets of matrices C such that $F_C(A) = F_C(\overline{A})$ for all $A \in M_n^{(k)}$, for various k. Some information on these sets is contained in the next proposition.

Proposition 5.3 *Let $\psi_n^{(k)}$ be the set of matrices $C \in M_n$ such that*

$$w_C(A) = w_C(\overline{A}) \quad \text{for all} \quad A \in M_n^{(k)}.$$

Then

$$\psi_n^{(n)} \subseteq \psi_n^{(n-1)} \subseteq \cdots \subseteq \psi_n^{(1)} = M_n. \tag{5.1}$$

(a) *Suppose C has rank at most k. Then $C \in \psi_n^{(k)}$ if and only if C is unitarily similar to $\mu\overline{C}$ for some $\mu \in \mathbf{T}$. Consequently, $\psi_n^{(n)}$ consists of those $C \in M_n$ such that C and $\mu\overline{C}$ are unitarily similar for some $\mu \in \mathbf{T}$.*

(b) *Assume $8k \le n$, and suppose C is unitarily similar to $(C_1 \otimes I_{4k}) \oplus C_2$ with $C_1 = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ such that $W(C_2) \subseteq W(C_1)$. Then $C \in \psi_n^{(k)}$.*

Part (b) illustrates that a complete characterization of the set $\psi_n^{(k)}$ (if $k < n$) may be not transparent.

Proof. The inclusions in (5.1) are trivial. To prove the equality $\psi_n^{(1)} = M_n$, fix $C \in M_n$, and let $A = xy^*$ be a rank one matrix, and let U be a unitary matrix. Define $\mu = (\overline{y^*x})/(y^*x)$ if $y^*x \ne 0$, and $\mu = 1$ otherwise. Then

$$\overline{y^*x} = y^*(\mu x) = (Uy)^*(U(\mu x)),$$

and therefore there exists a unitary V such that $V\overline{x} = \mu Ux$ and $V\overline{y} = Uy$. Thus,

$$\mathrm{tr}\,(CUAU^*) = \mathrm{tr}\,(CUxy^*U^*) = \mu\mathrm{tr}\,(CV\overline{x}\overline{y}^*V^*) = \mathrm{tr}\,(CV\overline{A}V^*),$$

and since U was an arbitrary unitary matrix, we have $w_C(A) \le w_C(\overline{A})$. The equality $w_C(A) = w_C(\overline{A})$ follows by reversing the roles of A and $\overline{A}$, and using (2.2) we obtain $\psi_n^{(1)} = M_n$.

For statement (a), the "if" part follows from Lemma 3.1. Conversely, suppose C has rank at most k. If $C \in \psi_n^{(k)}$, then $w_C(C^*) = w_C(C^t)$. Denote by $\|X\|_F = (\mathrm{tr}\,XX^*)^{1/2}$ the Frobenius norm on M_n. Then there exists a unitary U such that

$$\mathrm{tr}\,(CC^*) \le w_C(C^*) = w_C(C^t) = |\mathrm{tr}\,CUC^tU^*| \le \|C\|_F\|UC^tU^*\|_F = \mathrm{tr}\,(CC^*).$$

Using the equality case of Cauchy-Schwartz inequality, we see that $UC^tU^* = \mu C^*$ for some $\mu \in \mathbf{T}$. Hence C and $\mu\overline{C}$ is unitarily similar. The second statement in (a) is clear.

Next, we turn to statement (b). Assume that $C = (C_1 \otimes I_{4k}) \oplus C_2$. For simplicity, we assume that $a = 2$, i.e., $C_1 = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$. Suppose $A \in M_n^{(k)}$. Up to unitary similarity, we may assume that $A = A_1 \oplus 0_{n-2k}$, where A_1 is $2k \times 2k$. *We claim that $W_C(A) = W_{C_0}(A_0)$ and $W_C(\overline{A}) = W_{C_0}(\overline{A_0})$, where* $C_0 = C_1 \otimes I_{4k}$ and $A_0 = A_1 \oplus 0_{6k}$. Since C_0 is in block shift form, it will then follow by Lemma 4.1 that

$$W_C(A) = W_{C_0}(A_0) = W_{C_0}(\overline{A_0}) = W_C(\overline{A}),$$

and therefore also $w_C(A) = w_C(\overline{A})$.

To prove our claim, we first establish $W_{C_0}(A_0) \subseteq W_C(A)$. If $V \in M_{8k}$ and $z = \mathrm{tr}\,(VC_0V^*A_0) \in W_CC_0(A_0)$, then for $\tilde{V} = V \oplus I_{n-8k}$ we have $z = \mathrm{tr}\,(\tilde{V}C\tilde{V}^*A) \in W_C(A)$.

Next, we consider the reverse inclusion. Let $V \in M_n$ be unitary, and let

$$V^*CV = \begin{pmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{pmatrix}$$

with $C_{11} \in M_{2k}$ so that

$$\mathrm{tr}\,(CVAV^*) = \mathrm{tr}\,(V^*CVA) = \mathrm{tr}\,(C_{11}A_1). \tag{5.2}$$

Now, $W(C_{11}) \subseteq W(C) = W(C_1)$. By a result in [1] (see also [4]), $C_{11} = X^*(C_1 \otimes I_r)X$ for some positive integer $r > 0$ and some $2r \times 2k$ matrix X such that $X^*X = I_{2k}$.

If $r \le 4k$, there is a unitary matrix $V \in M_{8k}$ such that the first $2k$ rows of V^* have the form $[X^*|0_{2k,8k-2r}]$. Let $\tilde{C}_0 = (C_1 \otimes I_r) \oplus (C_1 \otimes I_{4k-r})$. Then

$$\mathrm{tr}\,(C_{11}A_1) = \mathrm{tr}\,(V^*\tilde{C}_0VA_0) \in W_{\tilde{C}_0}(A_0) = W_{C_0}(A_0),$$

where the last equality holds because $\tilde{C}_0$ and C_0 are unitarily similar.

Suppose $r > 4k$. Partition $X^* = [X_1^*|X_2^*]$, where each X_i^* is $2k \times r$. Let $U \in M_r$ be a unitary matrix such that the linear span of the first $4k$ rows of U^* contains all the rows of X_1^* and those of X_2^*. Then $X_i^*U = [Y_i^*|0]$, $i = 1, 2$, where Y_i^* is $2k \times 4k$. Thus,

$$\begin{aligned} C_{11} &= [X_1^*|X_2^*] \begin{pmatrix} 0_r & 2I_r \\ 0_r & 0_r \end{pmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \\ &= [X_1^*|X_2^*] \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & U^* \end{pmatrix} \begin{pmatrix} 0_r & 2I_r \\ 0_r & 0_r \end{pmatrix} \begin{pmatrix} U & 0 \\ 0 & U \end{pmatrix} \begin{pmatrix} U^* & 0 \\ 0 & U^* \end{pmatrix} \begin{bmatrix} X_1 \\ X_2 \end{bmatrix} \\ &= [Y_1^*|0|Y_2^*|0] \begin{pmatrix} 0_r & 2I_r \\ 0_r & 0_r \end{pmatrix} \begin{bmatrix} Y_1 \\ 0 \\ Y_2 \\ 0 \end{bmatrix} \\ &= [Y_1^*|Y_2^*] \begin{pmatrix} 0_{4k} & 2I_{4k} \\ 0_{4k} & 0_{4k} \end{pmatrix} \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix}. \end{aligned}$$

Note that $[Y_1^*|Y_2^*] \begin{bmatrix} Y_1 \\ Y_2 \end{bmatrix} = I_{2k}$. Suppose $R \in M_{8k}$ is unitary such that the first $2k$ rows of R^* equal $[Y_1^*|Y_2^*]$. Then by (5.2)

$$\mathrm{tr}\,(V^*CVA) = \mathrm{tr}\,(C_1A_1) = \mathrm{tr}\,(R^*C_0RA_0) \in W_{C_0}(A_0).$$

Hence $W_C(A) \subseteq W_{C_0}(A_0)$.

Combining the above arguments, we see that $W_{C_0}(A_0) = W_C(A)$. Similarly, one can prove that $W_{C_0}(\overline{A_0}) = W_C(\overline{A})$. Our claim is proved and the result follows. ∎

Proposition 5.4 *Let* $\Psi_n^{(k)}$ *be the set of matrices* $C \in M_n$ *such that*

$$W_C(A) = W_C(\overline{A}) \quad \textit{for all} \quad A \in M_n^{(k)}.$$

Then

$$\Psi_n^{(n)} \subseteq \Psi_n^{(n-1)} \subseteq \cdots \subseteq \Psi_n^{(1)}.$$

(a) *Suppose* C *has rank at most* k. *Then* $C \in \Psi_n^{(k)}$ *if and only if* C *is unitarily similar to a block shift matrix as well as unitarily similar to* $\mu\overline{C}$ *for some* $\mu \in \mathbf{T}$. *Consequently,* $\Psi_n^{(n)}$ *consists of those* $C \in M_n$ *such that* C *is unitarily similar to a block shift matrix as well as to* $\mu\overline{C}$ *for some* $\mu \in \mathbf{T}$.

(b) *Assume that* $8k \le n$, *and suppose* C *is unitarily similar to* $(C_1 \otimes I_{4k}) \oplus C_2$ *with* $C_1 = \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix}$ *such that* $W(C_2) \subseteq W(C_1)$. *Then* $C \in \Psi_n^{(k)}$.

Proof. The inclusion relation is clear. For statement (a), the "if' part follows from Lemmas 3.1 and 4.1. For the converse, the fact that C is unitarily similar to $\mu\overline{C}$ for some $\mu \in \mathbf{T}$ follows from Proposition 5.3 (a). Now, for any $\nu \in \mathbf{T}$,

$$\overline{\nu} W_C(C^*) = W_C(\overline{\nu} C^*) = W_C(\nu C^t) = \nu W_C(C^t).$$

Thus, $W_C(C^*) = \nu^2 W_C(C^t)$ for all $\nu \in \mathbf{T}$. Since $W_C(C^*)$ is star-shaped (see [3]), it is a circular disk centered at the origin. By Lemma 4.1, C is unitarily similar to a block shift matrix.

The proof of (b) is contained in that of Proposition 5.3 (b). ∎

Remark 5.5 Note that by the proof of Proposition 5.4, if $W_C(A) = W_C(\overline{A})$ for all $A \in M_n^{(k)}$ then $W_C(A)$ is a circular disk for all $A \in M_n^{(k)}$.

Remark 5.6 A characterization of matrices in the set $\Psi_n^{(k)}$ seems to be even more elusive than that of $\psi_n^{(k)}$. Even for $\Psi_n^{(1)}$ the situation is not as nice as for $\psi_n^{(1)} = M_n$. In fact, if $A \in M_n$ has rank 1, then A is unitarily similar to $\|A\|(qE_{1,1} + \sqrt{1-|q|^2}E_{1,2})$, for some $q \in \mathbb{C}$, $|q| \le 1$, and therefore $W_C(A) = \|A\| W_q(C)$, where

$$W_q(C) = \{qx^*Cx + \sqrt{1-|q|^2}x^*Cy : x, y \in \mathbb{C}^n, x^*x = 1 = y^*y, x^*y = 0\}$$

is the q-numerical range of C; see [18, 21, 13]. Moreover, it is known that

$$W_q(C) = \cup_{z \in W(C)} R(z),$$

where

$$R(z) = \Big\{ qz + \sqrt{1-|q|^2}\mu \in \mathbb{C} : |\mu|^2 + |z|^2 \le \|Ch\|^2$$
$$\text{for some } x \in \mathbb{C}^n \text{ with } (x^*x, x^*Cx) = (1, z) \Big\}.$$

Here $\|Cx\|$ is the Euclidean length of the vector Cx. By the above discussion and Remark 5.5 we see that $C \in \Psi_n^{(1)}$ *if the outer boundary of the set*

$$S_h = \{x^*Cx \,:\, x \in \mathbb{C}^n, \quad x^*x = 1, \quad \|Cx\| = h\}$$

is a circle or empty for any $h \ge 0$.

For example, if C is unitarily similar to a block shift matrix, or if C is unitarily similar to a matrix of the form

$$\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \oplus B, \qquad w(B) \le 1,$$

then C satisfies the above condition on the outer boundary, i.e., $C \in \Psi_n^{(1)}$.

We conclude the paper with an open problem.

Problem 5.7 *Obtain intrinsic characterizations of the classes* $\psi_n^{(k)}$ *and* $\Psi_n^{(k)}$ *in the general situation.*

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