Linear preservers of higher rank numerical ranges and radii

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Abstract

Structural theorems regarding linear preservers of the higher rank numerical ranges are proved for the real linear space of bounded selfadjoint operators or the complex linear space of bounded linear operators acting on a Hilbert space. It is shown that the linear preservers of rank k-numerical ranges must be of the standard form: unitary similarity or unitary similarity followed by transposition with respect to a fixed orthonormal basis. Furthermore, it is shown that a linear preserver of the rank k-numerical radius must be a unimodular scalar multiple of a linear preserver of the rank k-numerical range.

Key words: Linear preservers, higher rank numerical ranges, bounded operators, selfadjoint operators, unitary operators.

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1 Introduction and statement of results

Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on a complex Hilbert space $\mathcal{H}$. We identify $\mathcal{B}(\mathcal{H})$ with M_n, the algebra of $n \times n$ complex matrices, if $\mathcal{H}$ has

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dimension n. For a positive integer $k < \dim \mathcal{H}$, define the *rank-k numerical range* of $A \in \mathcal{B}(\mathcal{H})$ by

$$\Lambda_k(A) = \{\lambda \in \mathbb{C} : PAP = \lambda P \text{ for some rank } k \text{ orthogonal projection } P \in \mathcal{B}(\mathcal{H})\}.$$

Note that the cases when $\Lambda_k(A)$ is empty are not excluded.

The following proposition is clear.

Proposition 1.1 *Let $A \in \mathcal{B}(\mathcal{H})$ and k be a positive integer. The following conditions are equivalent for a given $\lambda \in \mathbb{C}$.*

(a) $\lambda \in \Lambda_k(A)$.

(b) *$\mathcal{H}$ has an orthonormal basis such that λI_k is the leading principal $k \times k$ submatrix of the operator matrix of A with respect to the basis.*

(c) *There is $X : \mathbb{C}^k \to \mathcal{H}$ such that $X^*X = I_k$ and $X^*AX = \lambda I_k$.*

We will often use the two other equivalent formulations of $\Lambda_k(A)$ in the above proposition in our discussion.

When $k = 1$, the rank k-numerical range reduces to the *classical numerical range* of A defined by

$$W(A) = \{\langle Au, u\rangle : u \in \mathcal{H},\ \langle u, u\rangle = 1\},$$

which is useful in studying operators and matrices; for example see [2]. Motivated by theory and applications, there are many generalizations of the numerical range, and there has been a great deal of interest in studying linear preservers of a given generalized numerical range, i.e., linear operators which leave invariant the given generalized numerical ranges, see [4].

The purpose of this paper is to characterize linear preservers of the rank k-numerical range. It is clear from the definition that if $U \in \mathcal{B}(\mathcal{H})$ is unitary then a mapping of the form

$$A \mapsto U^*AU \qquad \text{or} \qquad A \mapsto U^*A^tU,$$

where A^t is the transpose of $A \in \mathcal{B}(\mathcal{H})$ under a fixed orthonormal basis, will leave invariant the rank k-numerical range. We will prove that the converse is also true. In quantum computing, a change of bases for the states represented as trace one positive semidefinite operators correspond to a change of orthonormal bases and is achieved by a unitary similarity transforms. Similar comments apply to a change of bases for the measurement operators, quantum channels, etc. So, our results imply that linear preservers of rank k-numerical ranges are basically those operators corresponding to the change of state bases. In addition to $\mathcal{B}(\mathcal{H})$, we also obtain results for (real) linear preservers of the rank k-numerical range on the (real) linear space $\mathcal{S}(\mathcal{H})$ of bounded selfadjoint operators in $\mathcal{B}(\mathcal{H})$. If $\dim \mathcal{H}$ is finite, then for any $A \in \mathcal{B}(\mathcal{H})$ we have $\Lambda_k(A) = \overline{\Lambda_k(A)}$, where $\overline{S}$ denotes the closure of $S \subseteq \mathbb{C}$. But this may not be true if $\dim \mathcal{H}$ is infinite; see [5]. Our result also covers the linear preservers of the closure of the rank k-numerical range on $\mathcal{B}(\mathcal{H})$ or $\mathcal{S}(\mathcal{H})$. Here is the statement of our first main result.

Theorem 1.2 *Let $\mathcal{V} = \mathcal{S}(\mathcal{H})$ or $\mathcal{V} = \mathcal{B}(\mathcal{H})$. The following statements are equivalent for a surjective $\mathbb{F}$-linear map $\phi : \mathcal{V} \to \mathcal{V}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ depending on $\mathcal{V} = \mathcal{S}(\mathcal{H})$ or $\mathcal{V} = \mathcal{B}(\mathcal{H})$.*

(a) $\Lambda_k(A) = \Lambda_k(\phi(A))$ *for all* $A \in \mathcal{V}$.

(b) $\overline{\Lambda_k(A)} = \overline{\Lambda_k(\phi(A))}$ *for all* $A \in \mathcal{V}$.

(c) *There exists a unitary $U \in \mathcal{B}(\mathcal{H})$ such that*

  (1) $\phi(A) = U^*AU$ *for all* $A \in \mathcal{V}$, *or*

  (2) $\phi(A) = U^*A^tU$ *for all* $A \in \mathcal{V}$,

  where A^t is the transpose of A with respect to a fixed orthonormal basis for $\mathcal{H}$.

The surjective assumption on ϕ can be removed if $\dim \mathcal{H}$ *is finite.*

We also extend the definition of the *classical numerical radius*

$$r(A) = \sup\{|\mu| : \mu \in W(A)\}$$

to *rank k-numerical radius* defined by

$$r_k(A) = \sup\{|\mu| : \mu \in \Lambda_k(A)\}$$

with the convention that $r_k(A) = -\infty$ if $\Lambda_k(A) = \emptyset$, which may happen if $\dim \mathcal{H} \leq 3k-3$; see [1] and [6]. Clearly, if $\xi \in \mathbb{F}$ satisfies $|\xi| = 1$ and ϕ is a linear preserver of the rank k-numerical range on $\mathcal{V} = \mathcal{S}(\mathcal{H})$ or $\mathcal{B}(\mathcal{H})$, then $\xi\phi$ is a linear preserver of the rank k-numerical radius. It turns out that the converse is true as well, which resemble many existing results on preservers of generalized numerical ranges and radii; see [4]. Here is our result on rank k-numerical radius preservers.

Theorem 1.3 *Let $\mathcal{V} = \mathcal{S}(\mathcal{H})$ or $\mathcal{V} = \mathcal{B}(\mathcal{H})$. The following statements are equivalent for a surjective $\mathbb{F}$-linear map $\phi : \mathcal{V} \to \mathcal{V}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$ depending on $\mathcal{V} = \mathcal{S}(\mathcal{H})$ or $\mathcal{V} = \mathcal{B}(\mathcal{H})$.*

(a) $r_k(A) = r_k(\phi(A))$ *for all* $A \in \mathcal{V}$.

(b) *There exist a unitary $U \in \mathcal{B}(\mathcal{H})$ and $\xi \in \mathbb{F}$ with $|\xi| = 1$ such that*

(1) $\phi(A) = \xi U^* A U$ *for all* $A \in \mathcal{V}$, *or*

(2) $\phi(A) = \xi U^* A^t U$ *for all* $A \in \mathcal{V}$,

where A^t is the transpose of A with respect to a fixed orthonormal basis for $\mathcal{H}$.

The surjective assumption on ϕ can be removed if $\dim \mathcal{H}$ is finite.

We will give the proof of Theorem 1.2 for bounded selfadjoint operators in Section 2 and the proof of Theorem 1.2 for bounded operators in Section 3. The proof of Theorem 1.3 will be given in Section 4.

The following notation will be used in our discussion.

- $\mathrm{diag}\,(x_1, \ldots, x_m)$ denotes the $m \times m$ diagonal matrix with diagonal elements $x_1, \ldots, x_m$ (in that order);
- $\mathrm{rank}\,(A)$ is the rank of an operator $A \in \mathcal{B}(\mathcal{H})$;
- $\Re A = (A + A^*)/2$ and $\Im A = (A - A^*)/(2\mathrm{i})$ are the real part and imaginary part of $A \in \mathcal{B}(\mathcal{H})$;
- A^t is the transpose of $A \in \mathcal{B}(\mathcal{H})$ with respect to a fixed orthonormal basis.

If $\dim \mathcal{H} = n$, we will identify $\mathcal{S}(\mathcal{H})$ with the real linear space H_n of $n \times n$ Hermitian matrices. The eigenvalues of $A \in H_n$ will be denoted by $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$.

We will often use the facts about $\Lambda_k(A)$.

- $\Lambda_k(A) = \Lambda_k(A^t) = \Lambda_k(U^* A U)$ for any unitary $U \in \mathcal{B}(\mathcal{H})$.
- $\Lambda_k(\alpha A + \beta I) = \alpha \Lambda_k(A) + \beta$ for any $\alpha, \beta \in \mathbb{C}$.
- If $z \in \Lambda_k(A)$, then $\Re z \in \Lambda_k(\Re A)$ and $\Im z \in \Lambda_k(\Im A)$.

Since Theorems 1.2 and 1.3 are well known for $k = 1$ (for example, see [4]), we always assume that $k > 1$ in our discussion. In particular, $\dim \mathcal{H} \geq 3$.

2 Proof of Theorem 1.2 for bounded selfadjoint operators

We present the proof of Theorem 1.2 for bounded selfadjoint operators in this section. It is easy to determine $\overline{\Lambda_k(A)}$ as follows; see [6, 8].

Proposition 2.1 *Let $A \in \mathcal{S}(\mathcal{H})$. Suppose $\dim \mathcal{S}(\mathcal{H}) \geq 2k-1$. Then $\Lambda_k(A)$ is a non-empty convex subset of $\mathbb{R}$ such that*

$$\overline{\Lambda_k(A)} = [L_k(A), R_k(A)]$$

with

$$L_k(A) = \inf\{\lambda_1(X^*AX) : X^*X = I_k\} \quad \text{and} \quad R_k(A) = \sup\{\lambda_k(X^*AX) : X^*X = I_k\}.$$

In case $\dim \mathcal{H} = n$ is finite and A has eigenvalues $\lambda_1(A) \geq \cdots \geq \lambda_n(A)$, we have

$$\Lambda_k(A) = \overline{\Lambda_k(A)}, \quad L_k(A) = \lambda_{n-k+1}(A) \quad \text{and} \quad R_k(A) = \lambda_k(A).$$

If $\dim \mathcal{H} < 2k-1$, then either

(i) *$\lambda_k(A) < \lambda_{n-k+1}(A)$ and $\Lambda_k(A) = \emptyset$, or*

(ii) *$\lambda_k(A) = \lambda_{n-k+1}(A)$ and $\Lambda_k(A) = \{\lambda_k(A)\}$.*

To prove Theorem 1.2 for $\mathcal{V} = \mathcal{S}(\mathcal{H})$, note that the implications (c) $\Rightarrow$ (a) $\Rightarrow$ (b) are clear in Theorem 1.2. We focus on the proof of the implication (b) $\Rightarrow$ (c).

To achieve this, we will first show that ϕ is injective. Then it will be bijective in the finite dimensional case, and bijective under the surjective assumption of the theorem. We will then show that ϕ maps the set of positive semidefinite operators onto itself if $\dim \mathcal{H} \geq 2k$, and ϕ maps the set of operators in $\mathcal{S}(\mathcal{H})$ with rank $2(n-k)$ to matrices with rank at most $2(n-k)$ if $\dim \mathcal{H} = n < 2k$. We can then apply the following two lemmas; see [7], [3, 9], and also [12], [11, Chapters 2 and 3].

Lemma 2.2 *Let $\psi : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ be an invertible linear operator such that $\phi(I) = I$. Then $\psi(P) = P$, where $P \subseteq \mathcal{S}(\mathcal{H})$ is either the set of positive semidefinite operators or the set of positive definite invertible operators if and only if there is a unitary operator $S \in \mathcal{B}(\mathcal{H})$ such that ψ has the form*

$$(1) \quad \psi(A) = S^*AS \quad \forall \quad A \in \mathcal{S}(\mathcal{H}) \quad \text{or} \quad (2) \quad \psi(A) = S^*A^tS \quad \forall \quad A \in \mathcal{S}(\mathcal{H}).$$

Proof. The "if" part is obvious. For the "only if" part, note that for $A \in \mathcal{S}(\mathcal{H})$,

$$\inf W(A) = \sup\{t \in \mathbb{R} : A - tI \ \text{ is positive (semi)definite}\}$$

and

$$\sup W(A) = \inf\{t \in \mathbb{R} : A - tI \ \text{ is negative (semi)definite}\}.$$

Thus, the given assumption implies that $W(A)$ and $W(\psi(A))$ always have the same closure. The result then follows from [7, Theorem 2]. $\square$

Lemma 2.3 *Suppose* $1 \le r < n$ *and* $n \ge 3$*. Let* $\psi : H_n \to H_n$ *be an invertible linear operator. Then* ψ *maps the set of matrices with rank* r *to matrices of rank at most* r *if and only if there are* $\xi \in \{1, -1\}$ *and an invertible matrix* $S \in M_n$ *such that* ψ *has the form*

$$A \mapsto \xi S^*AS \quad \forall \ \ A \in H_n \qquad \text{or} \qquad A \mapsto \xi S^*A^tS \quad \forall \ \ A \in H_n.$$

If, in addition, $\psi(I) = I$*, then* $\xi = 1$ *and* S *is unitary.*

The next lemma and its proof take after [10, Lemma 2]. Let $\pi(A)$ and $\nu(A)$ is the number of positive and negative eigenvalues of a Hermitian matrix A respectively, counted with multiplicities.

Lemma 2.4 *Let* r, s *be positive integers such that* $r + s < n$*. Let* $\psi : H_n \to H_n$ *be a linear map on* H_n *with the following property:*

$$\operatorname{rank}(\psi(A)) \le r + s \quad \text{whenever} \quad A \in H_n \quad \text{satisfies} \quad \pi(A) \le r \quad \text{and} \quad \nu(A) \le s. \tag{2.1}$$

Then $\operatorname{rank}(A) \le r + s$ *implies* $\operatorname{rank}(\psi(A)) \le r + s$*.*

Proof. Let $m = r + s$. Since any $A \in H_n$ with $\operatorname{rank}(A) < m$ can be approximated with Hermitian matrices of rank m, it clearly suffices to show that $\operatorname{rank}(\psi(A)) \le m$ whenever $A \in H_n$ and $\operatorname{rank}(A) = m$. Suppose first $A \in H_n$, $\pi(A) = r + 1$ and $\nu(A) = s - 1$. Then there exists an invertible S such that $A = S^*DS$, where

$$D = \operatorname{diag}(a_1, \ldots, a_r, a_{r+1}, -b_1, -b_2, \ldots, -b_{s-1}, 0, \ldots, 0),$$

and $a_1, \ldots, a_r, a_{r+1}, b_1, \ldots, b_{s-1}$ are positive. Let

$$D_\epsilon = \operatorname{diag}(a_1, \ldots, a_r, \epsilon, -b_1, -b_2, \ldots, -b_{s-1}, 0, \ldots, 0), \quad \epsilon \in \mathbb{R},$$

$B_\epsilon = S^* D_\epsilon S$, $C_\epsilon = \psi(B_\epsilon)$. Then, for any $\epsilon < 0$, we have $\pi(B_\epsilon) = r$, $\nu(B_\epsilon) = s$, therefore $\operatorname{rank}(C_\epsilon) \leq m$. Hence every $(m+1) \times (m+1)$ minor of C_ϵ which is a polynomial on ϵ vanishes for all $\epsilon < 0$. Therefore every such minor vanishes for all real ϵ, in particular $\operatorname{rank} C_{a_{r+1}} \leq m$. But

$$\psi(A) = \psi(S^* D_{a_{r+1}} S) = \psi(B_{a_{r+1}}) = C_{a_{r+1}},$$

so $\operatorname{rank}(\psi(A)) \leq m$. Repeating the process one obtains $\operatorname{rank}(\psi(A)) \leq m$ as soon as $\operatorname{rank}(A) = m$ and $\pi(A) > r$. Analogously, we conclude that $\operatorname{rank}(\psi(A)) \leq m$ whenever $\operatorname{rank}(A) = m$ and $\nu(A) > s$. □

Next, we establish several lemmas characterizing some special operators in $\mathcal{S}(\mathcal{H})$ in terms of the higher rank numerical range. The next lemma will also be useful for discussion in Section 4.

Lemma 2.5 *Suppose $A \in \mathcal{S}(\mathcal{H})$ satisfies $r_k(A) = 0$. If $A \neq 0$ then there is $B \in \mathcal{S}(\mathcal{H})$ with $r_k(B) \in \{-\infty, 0\}$ such that*

$$r_k(A + B) > 0. \tag{2.2}$$

Proof. Since $A \neq 0$, there is a unit vector $u \in \mathcal{H}$ such that $\langle Au, u \rangle = \gamma \neq 0$. We may assume that $\gamma > 0$. Otherwise, consider $-A$ instead of A.

Suppose $\dim \mathcal{H} \geq 2k$. Let $\mathcal{H}_1$ be a $2k$ dimensional subspace of $\mathcal{H}$ containing u, and let A have operator matrix

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}$$

with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$. We may further assume that $A_{11} = \operatorname{diag}(a_1, \ldots, a_{2k})$ with $a_1 \geq \cdots \geq a_{2k}$ by choosing a suitable orthonormal basis for $\mathcal{H}_1$. Since $\Lambda_k(A_{11}) \subseteq \Lambda_k(A) = \{0\}$, we see that $a_k = 0 = a_{k+1}$. Since $u \in \mathcal{H}_1$, we see that $a_1 \geq \gamma > 0$. Let $B \in \mathcal{S}(\mathcal{H})$ have operator matrix $B_{11} \oplus 0_{\mathcal{H}_1^\perp}$ with

$$B_{11} = \operatorname{diag}(a_1 - a_1, a_1 - a_2, \ldots, a_1 - a_k) \oplus 0_k.$$

Then B is positive semidefinite with rank at most $k-1$. By Proposition 2.1, $\Lambda_k(B) = \{0\}$ so that $r_k(B) = 0$. But $a_1 I_k$ is the leading principal submatrix of $A_{11} + B_{11}$ so that $a_1 \in \Lambda_k(A + B)$. Hence, $r_k(A + B) \geq a_1 > 0 = r_k(B)$.

Suppose $\dim \mathcal{H} = n < 2k$. With a suitable orthonormal basis, we may assume that $A = \operatorname{diag}(a_1, \ldots, a_n)$ with $a_1 \geq \cdots \geq a_n$. Then $a_1 \geq \langle Au, u \rangle = \gamma > 0$. Let

$$B = \operatorname{diag}(0, a_1 - a_2, \ldots, a_1 - a_k, 0, \ldots, 0).$$

Then

$$\lambda_k(B) = 0 \le a_1 - a_{2k-n} = \lambda_{n-k+1}(B)$$

so that $r_k(B) \in \{-\infty, 0\}$ by Proposition 2.1, and $r_k(A+B) = a_1 > 0$. $\square$

Lemma 2.6 *Let $A \in \mathcal{S}(\mathcal{H})$ and $\alpha \in \mathbb{R}$. Then $A = \alpha I$ if and only if*

$$\Lambda_k(A+X) = \Lambda_k(X) + \alpha \qquad \forall \;\; X \in \mathcal{S}(\mathcal{H}), \tag{2.3}$$

or equivalently

$$\overline{\Lambda_k(A+X)} = \overline{\Lambda_k(X)} + \alpha \qquad \forall \;\; X \in \mathcal{S}(\mathcal{H}).$$

Proof. The "only if" part is clear from Proposition 2.1.

Assume (2.3) holds. Let $\widetilde{A} = A - \alpha I$. Then (2.3) implies that $r_k(\widetilde{A} + X) = r_k(X)$ for all $X \in \mathcal{S}(\mathcal{H})$. By Lemma 2.5, we see that $\widetilde{A} = 0$. Thus, $A = \alpha I$. $\square$

Lemma 2.7 *Suppose* $\dim \mathcal{H} \ge 2k-1$. *Then* $A \in \mathcal{S}(\mathcal{H})$ *with* $\inf \Lambda_k(A) \ge 0$ *is positive semidefinite if and only if*

$$\inf \Lambda_k(B) \le \inf \Lambda_k(A+B) \qquad \forall \;\; B \in \mathcal{S}(\mathcal{H}). \tag{2.4}$$

Proof. Let $A \in \mathcal{S}(\mathcal{H})$ be positive semidefinite. Suppose $B \in \mathcal{S}(\mathcal{H})$. Then for any $X : \mathbb{C}^k \to \mathcal{H}$ satisfying $X^*X = I_k$, we have

$$\lambda_1(X^*BX) \le \lambda_1(X^*(A+B)X)$$

by the well known properties of positive semidefinite operators. Thus, (2.4) holds.

Conversely, if A is not positive semidefinite, then there is a unit vector $u \in \mathcal{H}$ such that $\gamma := \langle Au, u\rangle < 0$. Following the argument in the proof of Lemma 2.5, there is a $2k-1$ dimensional subspace $\mathcal{H}_1$ of $\mathcal{H}$ containing the vector u so that A has operator matrix

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}$$

with respect to the decomposition $\mathcal{H} = \mathcal{H}_1 \oplus \mathcal{H}_1^\perp$. We may further assume that $A_{11} = \operatorname{diag}(a_1, \ldots, a_{2k-1})$ with $a_1 \ge \cdots \ge a_{2k-1}$ by choosing a suitable orthonormal basis for $\mathcal{H}_1$. Since $\{a_k\} = \Lambda_k(A_{11}) \subseteq \Lambda_k(A) \subseteq [0, \infty)$, we see that $a_k \ge 0$. Since $u \in \mathcal{H}_1$, we see that $a_{2k-1} \le \gamma < 0$. Let $B \in \mathcal{S}(\mathcal{H})$ be given by the operator matrix $B_{11} \oplus 0_{\mathcal{H}_1^\perp}$ with

$B_{11} = 0_{k-1} \oplus -MI_{k-1} \oplus [0]$, where M satisfies $a_{2k-1} \geq a_k - M$. Then $\Lambda_k(B) = \{0\}$, so that $\inf \Lambda_k(B) = 0$. But

$$\inf \Lambda_k(A+B) \leq \inf \Lambda_k(A_{11} + B_{11}) = a_{2k-1} < 0,$$

which contradicts (2.4). □

Now, we are ready to present the

Proof of Theorem 1.2 for bounded selfadjoint operators

Suppose $\phi : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ is a linear map that satisfies condition (b) of Theorem 1.2, and that ϕ is surjective in case $\dim \mathcal{H}$ is infinite.

First, we show that ϕ is bijective. Under the assumption on ϕ, we need only prove that ϕ is injective.

Let $A \in \mathcal{S}(\mathcal{H})$ be such that $\phi(A) = 0$, so $\Lambda_k(A) = \Lambda_k(\phi(A)) = \{0\}$. Then for any $B \in \mathcal{S}(\mathcal{H})$ we have

$$r_k(B) = r_k(\phi(B)) = r_k(\phi(B) + \phi(A)) = r_k(\phi(A+B)) = r_k(A+B). \tag{2.5}$$

By Lemma 2.5, $A = 0$.

Now, ϕ is invertible. It is easy to see that ϕ^{-1} has the same property as ϕ has, i.e.,

$$\overline{\Lambda_k(B)} = \overline{\Lambda_k(\phi^{-1}(B))} \qquad \forall \ B \in \mathcal{S}(\mathcal{H}). \tag{2.6}$$

Next, we show that $\phi(I) = I$. To see this, note that $\overline{\Lambda_k(I+X)} = \overline{\Lambda_k(X)} + 1$ for all $X \in \mathcal{S}(\mathcal{H})$. It follows that

$$\overline{\Lambda_k(\phi(I)+Y)} = \overline{\Lambda_k(\phi(I+\phi^{-1}(Y)))} = \overline{\Lambda_k(I+\phi^{-1}(Y))} = 1 + \overline{\Lambda_k(\phi^{-1}(Y))} = 1 + \overline{\Lambda_k(Y)}$$

for all $Y \in \mathcal{S}(\mathcal{H})$. By Lemma 2.6, we see that $\phi(I) = I$.

We divide the rest of the proof into two cases.

Case 1. Suppose $\dim \mathcal{H} \geq 2k-1$. This ensures that $\Lambda_k(A) \neq \emptyset$ for every $A \in \mathcal{S}(\mathcal{H})$. Then we have

$$\inf \Lambda_k(A) = \inf \Lambda_k(\phi(A)) \qquad \forall \ A \in \mathcal{S}(\mathcal{H}). \tag{2.7}$$

Let $A \in \mathcal{S}(\mathcal{H})$ be positive semidefinite. Then (2.4) in Lemma 2.7 holds. It follows that

$$\inf \Lambda_k(Y) \leq \inf \Lambda_k(\phi(A)+Y) \qquad \forall \ Y \in \mathcal{S}(\mathcal{H}).$$

Thus, $\phi(A)$ is positive semidefinite. Applying the argument to ϕ^{-1}, we conclude that ϕ maps the set of positive semidefinite operators in $\mathcal{S}(\mathcal{H})$ onto itself. Since we have shown that $\phi(I) = I$, by Lemma 2.2 there exists a unitary S such that either $\phi(A) = S^*AS$ for all $A \in H_n$ or $\phi(A) = S^*A^tS$ for all $A \in \mathcal{S}(\mathcal{H})$.

Case 2 Suppose $2k - 1 > n \geq 3$. Identify $\mathcal{S}(\mathcal{H})$ with H_n. Consider the set

$$\Gamma_k := \{A \in H_n : \Lambda_k(A) = \{0\}\} = \{A \in H_n : \lambda_{n-k+1}(A) = \lambda_k(A) = 0\}.$$

Clearly, $\phi(\Gamma_k) \subseteq \Gamma_k$. Applying Lemma 2.4 with $r = s = n - k$, we have $\operatorname{rank} \phi(A) \leq 2(n-k)$ whenever $A \in H_n$ and $\operatorname{rank} A \leq 2(n-k)$. Since we have shown that $\phi(I) = I$, by Lemma 2.3 there exists a unitary S such that either $\phi(A) = S^*AS$ for all $A \in H_n$ or $\phi(A) = S^*A^tS$ for all $A \in H_n$.

This concludes the proof of (b) $\Rightarrow$ (c). $\square$

3 Proof of Theorem 1.2 for bounded operators

Similar to the discussion at the beginning of Section 2, we need only prove the implication (b) $\Rightarrow$ (c) for Theorem 1.2 when $\mathcal{V} = \mathcal{B}(\mathcal{H})$.

So, we assume that $\phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ is linear and satisfies condition (a) in Theorem 1.2.

Our strategy is to show that ϕ is invertible. If $\dim \mathcal{H} \geq 2k$, we will show that $\phi(\mathcal{S}(\mathcal{H})) \subseteq \mathcal{S}(\mathcal{H})$. If $\dim \mathcal{H} < 2k$, we show that there is an invertible linear map $\psi : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ associated with ϕ satisfying $\Lambda_k(\psi(A)) = \Lambda_k(A)$ whenever $A \in \mathcal{S}(\mathcal{H})$ with nonempty $\Lambda_k(A)$. In both cases, we will then be able to use the results in Section 2 to obtain the structure of ϕ.

By the results in [5, 8], we have the following general description of $\Lambda_k(A)$ for $A \in \mathcal{B}(\mathcal{H})$, which implies the convexity of $\Lambda_k(A)$ established in [1, 13].

Proposition 3.1 *Let $A \in \mathcal{B}(\mathcal{H})$. For $t \in [0, 2\pi)$, let*

$$R_{k,t}(A) = R_k((e^{\mathrm{i}t}A + e^{-\mathrm{i}t}A^*)/2),$$

where for $H \in \mathcal{S}(\mathcal{H})$

$$R_k(H) = \sup\{\lambda_k(X^*HX) : X^*X = I_k\}.$$

Then $\Lambda_k(A)$ is a bounded convex set such that

$$\overline{\Lambda_k(A)} = \{\mu \in \mathbb{C} : \Re(e^{\mathrm{i}t}\mu) \leq R_{k,t}(A) \ \ \forall \ \ t \in [0, 2\pi)\}.$$

We begin with the following lemma which will also be useful in Section 4.

Lemma 3.2 *Let $A \in \mathcal{B}(\mathcal{H})$ be such that $r_k(A) = 0$. If $A \neq 0$, then there is $B \in \mathcal{B}(\mathcal{H})$ with $r_k(B) \in \{-\infty, 0\}$ such that*

$$r_k(A + B) > 0. \tag{3.1}$$

Proof. Suppose $A \in \mathcal{B}(\mathcal{H})$ with $r_k(A) = 0$. Then $\Lambda_k(A) = \{0\}$ is non-empty. Suppose $A \in \mathcal{S}(\mathcal{H})$. The result follows from Lemma 2.5. So, assume $\Im A \neq 0$.

If $r_k(\Re A) > 0$, let $B = \Re A - \mathrm{i}\Im A$ so that

$$\Lambda_k(B) = \{x - \mathrm{i}y : x, y \in \mathbb{R}, x + \mathrm{i}y \in \Lambda_k(A)\}.$$

Then $r_k(B) = r_k(A) = 0$ and $r_k(A + B) = 2r_k(\Re A) > 0$.

Suppose $r_k(\Re A) = 0$ so that $\Lambda_k(\Re A) = \{0\}$. By Lemma 2.5, there is $C \in \mathcal{S}(\mathcal{H})$ such that $r_k(C) \in \{-\infty, 0\}$ and $r_k(\Im A + C) > 0$. Let $B = -\Re A + \mathrm{i}C$. Then

$$\Lambda_k(B) \subseteq \{x + \mathrm{i}y : x \in \Lambda_k(\Re A), y \in \Lambda_k(C)\}$$

so that $r_k(A + B) = r_k(\Im A + C) > 0 \geq r_k(B)$. □

Lemma 3.3 *Let $A \in M_n$ with $H = \Re A$ and $G = \Im A$. Assume $n \geq 2k$. Then $\Lambda_k(A)$ is a nondegenerate line segment in $\mathbb{R}$ if and only if $\Lambda_k(A)$ contains at least two distinct points and $\Lambda_k(G) = \{0\}$, i.e., $\lambda_k(G) = 0 = \lambda_{n-k+1}(G)$.*

Note that the hypothesis $n \geq 2k$ ensures that there exists $A \in \mathcal{B}(\mathcal{H})$ such that $\Lambda_k(A)$ has at least two points.

Proof. The "if" part is clear by Proposition 3.1.

We consider the "only if"' part. Assume $\Lambda_k(A) = [a, b] \subseteq \mathbb{R}$ with $a < b$ but that $\Lambda_k(G) \neq 0$. We may replace A by $\alpha A + \beta I$ for suitable $\alpha, \beta \in \mathbb{R}$ and assume that $\Lambda_k(A) = [-1, 1]$ and additionally $\lambda_k(-G) = d > 0$. We establish a contradiction by constructing a pure imaginary number $\alpha = -\mathrm{i}m$ for $m > 0$ in the range. It suits this purpose to only consider $\theta \in [0, \pi]$, because $0 \in \Lambda_k(A)$ implies

$$m \sin(\theta) < 0 \leq \lambda_k(H \cos(\theta) - G \sin(\theta)) \qquad \forall \;\; \theta \in (\pi, 2\pi).$$

Since $\lambda_k(H \cos(\theta) - G \sin(\theta))$ depends continuously on θ, for $\epsilon = d/2$, there exists $\delta > 0$ such that

$$|\lambda_k(H \cos(\theta) - G \sin(\theta)) - d| < d/2$$

whenever $|\theta - \pi/2| < \delta$. So

$$\lambda_k(H\cos(\theta) - G\sin(\theta)) > d/2 \quad \forall \;\; \theta \in (\pi/2 - \delta, \pi/2 + \delta).$$

Then for $\theta \in [0, \pi/2 - \delta]$, since $1 \in \Lambda_k(A)$ we have

$$0 < \cos(\pi/2 - \delta) \leq \cos(\theta) \leq \lambda_k(H\cos(\theta) - G\sin(\theta)).$$

Similarly, since $-1 \in \Lambda_k(A)$ we have

$$0 < \cos(\pi/2 - \delta) = -\cos(\pi/2 + \delta) \leq \lambda_k(H\cos(\theta) - G\sin(\theta))$$

for $\theta \in [\pi/2 + \delta, \pi]$. Now let $m = \min(d/2, \cos(\pi/2 - \delta))$. Then for $\alpha = -\mathrm{i}m$, we have

$$\sin(\theta)m \leq m \leq \lambda_k(H\cos(\theta) - G\sin(\theta)) \qquad \forall \;\; \theta \in [0, \pi],$$

thus $\alpha \in \Lambda_k(A)$ which is a contradiction. So $\Lambda_k(G) = 0$. $\square$

Lemma 3.4 *Suppose $A \in \mathcal{B}(\mathcal{H})$ where $\dim \mathcal{H} \geq 2k$. Then A is selfadjoint if and only if $\Lambda_k(A) \neq \emptyset$ and*

$$\Lambda_k(A+B) \subseteq \mathbb{R} \;\text{ whenever } B \in \mathcal{B}(\mathcal{H}) \text{ and } \overline{\Lambda_k(B)} \text{ is a nondegenerate line segment in } \mathbb{R}. \tag{3.2}$$

Proof. Suppose $A \in \mathcal{S}(\mathcal{H})$. Then $\Lambda_k(A)$ is non-empty subset of $\mathbb{R}$ by Proposition 2.1. Assume $B \in \mathcal{B}(\mathcal{H})$ is such that $\overline{\Lambda_k(B)}$ is a non-degenerate line segment in $\mathbb{R}$, and let $\mu\Lambda_k(A + B)$. Select two distinct points b_1, b_2 in $\Lambda_k(B)$, and let $X, Y, Z : \mathbb{C}^k \to \mathcal{H}$ be such that $X^*X = Y^*Y = Z^*Z = I_k$ and

$$X^*(A + B)X = \mu I_k, \quad Y^*BY = b_1 I_k, \quad Z^*BZ = b_2 I_k.$$

Now, let $V : C^m \to \mathcal{H}$ be such that $V^*V = I_m$ and the range space of V contains the range spaces of X, Y, Z. Here m is a suitable integer not exceeding $3k$. Then

$$\Lambda_k(V^*BV) \subseteq \Lambda_k(B) \subseteq \mathbb{R},$$

and $b_1, b_2 \in \Lambda_k(V^*BV)$, so by Lemma 3.3,

$$\lambda_k(\Im(V^*BV)) = 0 = \lambda_{n-k+1}(\Im(V^*BV)).$$

Also, $\mu \in \Lambda_k(V^*(A+B)V)$. Now, $A+B = (A+\Re B) + \mathrm{i}\Im B$, and

$$\Im\mu \in \Lambda_k(\Im(V^*(A+B)V)) \subseteq \Lambda_k(\Im(V^*BV)) = \{0\}.$$

Hence, $\Lambda_k(A+B) \subseteq \mathbb{R}$.

We establish the converse by proving the contra-positive. Suppose (3.2) holds but $\Im A \neq 0$. Then there is $Z : \mathbb{C}^{2k} \to \mathcal{H}$ such that $Z^*Z = I_{2k}$ and the matrix $\widetilde{A} := Z^*AZ$ has the properties $\Im\widetilde{A} \neq 0$ and $\Lambda_k(\widetilde{A})$ is a non-empty subset of $\Lambda_k(A)$. Let $\mathcal{H}_1$ be the range space of Z. Then we may assume that the operator matrix of A with respect to the decomposition $\mathcal{H}_1 \oplus \mathcal{H}_1^\perp$ has the form $\begin{bmatrix} \widetilde{A} & * \\ * & * \end{bmatrix}$. We will construct $B \in \mathcal{B}(\mathcal{H})$ with operator matrix $\begin{bmatrix} \widetilde{B} & 0 \\ 0 & 0 \end{bmatrix}$ such that $\widetilde{B} \in M_{2k}$ satisfies

$$[-1,1] \subseteq \Lambda_k(\widetilde{B}) = \Lambda_k(B) \quad \text{and} \quad \mu \in \Lambda_k(\widetilde{A}+\widetilde{B}) \subseteq \Lambda_k(A+B)$$

for some $\mu \in \mathbb{C} \setminus \mathbb{R}$. Then we get a contradiction.

For notational simplicity, we let $\dim \mathcal{H}_1 = 2k = n$. First, we verify that $\Lambda_k(\widetilde{A})$ is real. Indeed, if (up to a unitary similarity) $\widetilde{A} = \begin{bmatrix} wI_k & * \\ * & * \end{bmatrix}$ with nonreal w, then letting $\widetilde{B} = I_k \oplus (-I_k)$ we obtain a contradiction with (3.2).

Assume that X is $n \times k$ such that $X^*X = I_k$ and $X^*\widetilde{A}X = aI_k$ with $a \in \mathbb{R}$. Since $\Im\widetilde{A} \neq 0$, we can append a column x_0 to X to obtain a matrix $\widehat{X} = [X|x_0]$ so that

$$\widehat{A} := \widehat{X}^*\widetilde{A}\widehat{X} = H + \mathrm{i}G = \begin{bmatrix} aI_k & x \\ y^* & z \end{bmatrix}, \qquad H = \Re\widehat{A}, \quad G = \Im\widehat{A},$$

so that $G \neq 0$. Evidently, G can only have nonzero entries in the last row and last column. Let $U \in M_{k+1}$ be unitary having the form $U_1 \oplus [1]$ such that $U^*HU = H$ and U^*GU only have nonzero elements at the $(k,k), (k,k+1), (k+1,k)$ and $(k+1,k+1)$ entries. We can further find a unitary $V \in M_{k+1}$ have the form $I_{k-1} \oplus V_1$ such that V^*U^*GUV is a diagonal matrix with nonzero (k,k) entry equal to $g \in \mathbb{R}$. Then

$$V^*U^*\widehat{A}UV = \begin{bmatrix} aI_{k-1} & A_{12} \\ A_{12}^* & A_{22} \end{bmatrix}$$

with $A_{22} \in M_2$. Applying a unitary similarity to $\widetilde{A}$, we may assume that $V^*U^*\widehat{A}UV = (a_{ij})$ is the $(k+1) \times (k+1)$ upper left corner of the matrix $\widetilde{A}$. Let

$$\widetilde{B} = \begin{bmatrix} B_{11} & I_k \\ I_k & -B_{11} \end{bmatrix}, \qquad \text{where} \qquad B_{11} = \begin{bmatrix} \mathrm{i}gI_{k-1} & -v \\ -v^* & -\Re(a_{kk}) + a \end{bmatrix}$$

with $v = (a_{1k}, a_{2k}, \ldots, a_{k-1,k})^t$. It follows that the leading $k \times k$ principal submatrix of $V^*U^*\widehat{A}UV + \widetilde{B}$ is $(a + \mathrm{i}g)I_k$, and hence $a + \mathrm{i}g \in \Lambda_k(\widetilde{A} + \widetilde{B})$. Note that

$$\Im\widetilde{B} = gI_{k-1} \oplus [0] \oplus (-g)I_{k-1} \oplus [0], \quad \text{hence} \ \ \Lambda_k(\Im\widetilde{B}) = \{0\}. \tag{3.3}$$

Furthermore, if

$$R = \frac{1}{\sqrt{2}} \begin{bmatrix} I_k & I_k \\ I_k & -I_k \end{bmatrix},$$

then

$$R^*\widetilde{B}R = \begin{bmatrix} I_k & B_{11} \\ B_{11} & -I_k \end{bmatrix}.$$

Thus, $[-1, 1] \subseteq \Lambda_k(\widetilde{B})$. Hence $\Lambda_k(\widetilde{B}) \subseteq \mathbb{R}$ in view of (3.3) and Lemma 3.3. So, we have $\widetilde{B} \in M_n$ such that $\Lambda_k(\widetilde{B})$ is a non-degenerate line segment, and $\Lambda_k(\widetilde{A} + \widetilde{B})$ contains $a + \mathrm{i}g$. Thus, $\Lambda_k(\widetilde{A} + \widetilde{B}) \not\subseteq \mathbb{R}$, and the desired result follows. □

Lemma 3.5 *Suppose* $\{A_1, \ldots, A_{n^2}\}$ *is a basis for* M_n. *Then except for finitely many* $\gamma \in \mathbb{R}$ *the set* $\{\Re A_j + \gamma\Im A_j : 1 \le j \le n^2\}$ *is a basis for* H_n.

Proof. Write $A_j = H_j + iG_j$ for $j = 1, \ldots, n^2$ with $H_j, G_j \in H_n$. We identify the $n \times n$ Hermitian matrices H_j and G_j with vectors u_j and v_j in $\mathbb{R}^{n^2}$. Then A_j can be identified with the vector $u_j + \mathrm{i}v_j$ in $\mathbb{C}^{n^2}$. Define U and V by the $n^2 \times n^2$ matrices with u_j and v_j as the jth columns, respectively. Since $\{A_1, \ldots, A_{n^2}\}$ is a basis for M_n, $\{u_1 + \mathrm{i}v_1, \ldots, u_{n^2} + \mathrm{i}v_{n^2}\}$ is a basis in $\mathbb{C}^{n^2}$. In particular, we have $\det(U + \mathrm{i}V) \neq 0$. Then the set $\{H_j + \gamma G_j : 1 \le j \le n^2\}$ is a basis for H_n if and only if $p(\gamma) := \det(U + \gamma V) \neq 0$. Note that $p(\alpha)$ is a not identically zero polynomial of α in $\mathbb{C}$ with degree at most n^2. The result follows. □

We are now ready to present the

Proof of Theorem 1.2 for bounded operators

We need only to prove the implication (b) $\Rightarrow$ (c). Similarly to the selfadjoint case, we can show that ϕ is bijective using Lemma 3.2.

Now, ϕ is invertible. It is easy to see that ϕ^{-1} has the same property as ϕ has, i.e.,

$$\overline{\Lambda_k(B)} = \overline{\Lambda_k(\phi^{-1}(B))} \qquad \forall \ \ B \in \mathcal{B}(\mathcal{H}).$$

We consider two cases.

Case 1. Assume first $\dim \mathcal{H} \geq 2k$. We show that $\phi(\mathcal{S}(\mathcal{H})) \subseteq \mathcal{S}(\mathcal{H})$. Let $A \in \mathcal{S}(\mathcal{H})$. Then for every $B \in \mathcal{B}(\mathcal{H})$ such that $\overline{\Lambda_k(B)} = \overline{\Lambda_k(\phi^{-1}(B))}$ is a nondegenerate real interval in $\mathbb{R}$, by Lemma 3.3 we have $\Lambda_k(\Im(\phi^{-1}(B))) = \{0\}$, and moreover

$$\overline{\Lambda_k(\phi(A) + B)} = \overline{\Lambda_k(A + \phi^{-1}(B))}.$$

Since

$$A + \phi^{-1}(B) = A + \Re(\phi^{-1}(B)) + \mathrm{i}\Im(\phi^{-1}(B)),$$

by Proposition 3.1 we have

$$\overline{\Lambda_k(\phi(A) + B)} = \overline{\Lambda_k(A + \phi^{-1}(B))} \subseteq \mathbb{R}.$$

Also, $\overline{\Lambda_k(\phi(A))} = \overline{\Lambda_k(A)} \neq \emptyset$. By Lemma 3.4 we conclude that $\phi(A) \in \mathcal{S}(\mathcal{H})$. Now, we can define $\psi : \mathcal{S}(\mathcal{H}) \to \mathcal{S}(\mathcal{H})$ to be the restriction of ϕ onto $\mathcal{S}(\mathcal{H})$, i.e. $\psi(A) = \phi(A)$ for all $A \in \mathcal{S}(\mathcal{H})$, which is linear and preserves the rank k numerical range. Then by Theorem 1.2 for the selfadjoint case, there is a unitary $U \in \mathcal{B}(\mathcal{H})$ such that either

(1) $\phi(A) = U^*AU$ for all $A \in \mathcal{S}(\mathcal{H})$, or (2) $\phi(A) = U^*A^tU$ for all $A \in \mathcal{S}(\mathcal{H})$.

Since $\phi(H + \mathrm{i}G) = \phi(H) + \mathrm{i}\phi(G)$ for any $H, G \in \mathcal{S}(\mathcal{H})$, we see that ϕ has the standard form.

Case 2 Assume $\dim \mathcal{H} = n < 2k$. We identify $\mathcal{B}(\mathcal{H})$ as M_n. Suppose $\mathcal{B} = \{A_1, \ldots, A_{n^2}\}$ is a basis for H_n. Then $\{\phi(A_j) : 1 \leq j \leq n^2\}$ is a basis for M_n. By Lemma 3.5, except for a finite number of γ, the set

$$\mathcal{B}_\gamma = \{\Re\phi(A_j) + \gamma\Im\phi(A_j) : 1 \leq j \leq n^2\}$$

is a basis for H_n. Hence the real linear map $\psi_\gamma : H_n \to H_n$ defined by

$$\psi_\gamma(A) = \Re\phi(A) + \gamma\Im\phi(A),$$

maps the basis $\mathcal{B}$ to the basis $\mathcal{B}_\gamma$. Thus, ψ_γ is invertible.

Moreover, if $A \in H_n$ satisfies $\Lambda_k(A) = \{\alpha\}$, then $\Lambda_k(\phi(A)) = \{\alpha\}$. Thus, there is an $n \times k$ matrix X satisfying $X^*X = I_k$ and $X^*\phi(A)X = \alpha I_k$. As a result,

$$X^*\Re\phi(A)X = \alpha I_k \qquad \text{and} \qquad X^*\Im\phi(A)X = 0_k.$$

It follows that $X^*(\psi_\gamma(A))X = \alpha I_k$. Since $n < 2k$, $\Lambda_k(\psi_\gamma(A))$ must be a singleton and equal to $\{\alpha\}$. In particular, $\psi_\gamma(\Gamma_k) \subseteq \Gamma_k$, where

$$\Gamma_k := \{A \in H_n : \Lambda_k(A) = \{0\}\} = \{A \in H_n : \lambda_{n-k+1}(A) = \lambda_k(A) = 0\}.$$

Following the proof of Case 2 of the selfadjoint case, we conclude that ψ_γ has the standard form

$$A \mapsto U_\gamma^* A U_\gamma \quad \forall \;\; A \in H_n \qquad \text{or} \qquad A \mapsto U_\gamma^* A^t U_\gamma \quad \forall \;\; A \in H_n$$

for some unitary $U_\gamma \in M_n$.

Next, we claim that $\Im\phi(A) = 0$ for all $A \in \mathcal{S}(\mathcal{H})$. If it is not true and if there is $B \in \mathcal{S}(\mathcal{H})$ such that $\Im\phi(B) \neq 0$, then there exists a sufficiently large $\gamma > 0$ so that

$$\|\psi_\gamma(B)\| = \|\Re\phi(B) + \gamma\Im\phi(B)\| > \|B\|,$$

which contradicts the fact that ψ_γ is in standard form and will leave invariant the norms of matrices in H_n.

By the above argument, we see that there is a unitary $U \in M_n$ such that either

(1) $\phi(A) = U^*AU$ for all $A \in H_n$, or (2) $\phi(A) = U^*A^tU$ for all $A \in H_n$.

Since $\phi(H + \mathrm{i}G) = \phi(H) + \mathrm{i}\phi(G)$ for any $H, G \in H_n$, we see that ϕ has the standard form. $\square$

4 Proof of Theorem 1.3

The purpose of this section is to prove Theorem 1.3.

The implication of (b) $\Rightarrow$ (a) is clear, since $r_k(\xi A) = |\xi| r_k(A)$ for all $A \in \mathcal{V}$ and all $\xi \in \mathbb{F}$; by convention, $|\xi|(-\infty) = -\infty$ for all $\xi \in \mathbb{F}$.

We focus on the converse. So, we assume the general statement that $\phi : \mathcal{V} \to \mathcal{V}$ is a linear map with $r_k(A) = r_k(\phi(A))$ for all $A \in \mathcal{V}$.

The key step is to show that $\phi(I) = \xi I$ for some $\xi \in \mathbb{C}$. We can then show that $\xi^{-1}\phi$ will be a linear preserver of the rank k-numerical range so that Theorem 1.2 applies.

We need three lemmas to prove Theorem 1.3.

Lemma 4.1 *Let $X = X_1 \oplus 0_{\mathcal{G}}$, where $X_1 \in M_k$ and $\mathcal{G}$ is a Hilbert space of dimension at least $k-1$. If $0 \in W(X_1)$, then $\Lambda_k(X) = \{0\}$.*

Proof. Write $X = H + \mathrm{i}G$, $H = H^*$, $G = G^*$. Then $H = H_1 \oplus 0_{\mathcal{G}}$, where $H_1 = H_1^* \in M_k$. Also, zero belongs to the numerical range of H_1, so H_1 cannot be positive definite or negative definite. Thus, H_1 has at most $k-1$ positive eigenvalues and at most $k-1$ negative eigenvalues. So $\Lambda_k(H) = \{0\}$. Similarly, $\Lambda_k(G) = \{0\}$.

Therefore $\Lambda_k(X) \subseteq \{0\}$. But it is easy to see that $\Lambda_k(X)$ is nonempty, and we are done. $\square$

Lemma 4.2 *Let $A \in \mathcal{V}$ satisfy $r_k(A) > 0$. Then following statements are equivalent.*

(a) *A is a scalar operator.*

(b) *For every $B \in \mathcal{V}$ such that $\Lambda_k(B) \neq \emptyset$ we have*

$$r_k(A) + r_k(B) = \sup\{r_k(\xi A + B) : |\xi| = 1\}. \tag{4.1}$$

Proof. Note that for the case $\mathcal{V} = \mathcal{B}(\mathcal{H})$, our unimodular coefficients ξ in (4.1) can be drawn from the complex unit circle, whereas in the $\mathcal{V} = \mathcal{S}(\mathcal{H})$ case $\xi = \pm 1$.

To prove (a) $\Rightarrow$ (b), suppose $A = \mu I \in \mathcal{V}$ and let $B \in \mathcal{V}$ be such that $\Lambda_k(B) \neq \emptyset$. Since

$$\Lambda_k(\xi\mu I + B) = \Lambda_k(B) + \xi\mu,$$

we have $r_k(\xi A + B) \leq |\mu| + r_k(B)$, hence

$$\sup\{r_k(\xi A + B) : |\xi| = 1\} \leq r_k(A) + r_k(B). \tag{4.2}$$

Moreover, if $\{\nu_m\}$ is a sequence in $\Lambda_k(B)$ converging to ν such that $|\nu| = r_k(B)$, we can choose ξ_0 such that $|\xi| = 1$ ($\xi = \pm 1$ if $\mathcal{V} = \mathcal{S}(\mathcal{H})$) and

$$|\xi_0\mu + \nu| = |\mu| + |\nu| = r_k(A) + r_k(B).$$

Then $\{\xi_0\mu + \nu_m\}$ is a sequence in $\Lambda_k(\xi_0\mu I + B)$ converging to $\xi_0\mu + \nu$. (In the selfadjoint case, this reduces to matching the sign of μ to that of ν by multiplying by -1 if necessary.) Thus,

$$r_k(A) + r_k(B) \leq r_k(\xi_0 A + B) \leq \sup\{r_k(\xi A + B) : |\xi| = 1\}. \tag{4.3}$$

To prove the converse, assume (b) holds. We consider two cases.

Case 1. Suppose $\dim \mathcal{H} \geq 2k - 1$. We **claim** that $|\langle Ax, x\rangle| = r_k(A) = \gamma$ for any unit vector $x \in \mathcal{H}$.

Assume first that there is a unit vector $x \in \mathcal{H}$ satisfying $|\langle Ax, x\rangle| > \gamma$. Let $X : \mathbb{C}^{2k-1} \to \mathcal{H}$ be such that $X^*X = I_{2k-1}$ and x belongs to the range space of X, denoted by $\mathcal{H}_1$. Since $x \in \mathcal{H}_1$, there is a unit vector $y \in \mathcal{H}_1$ such that

$$|\langle Ay, y\rangle| = r(A_{11}) \geq |\langle Ax, x\rangle| > \gamma.$$

We may replace A by $e^{\mathrm{i}t}A$ for a suitable $t \in [0, 2\pi)$ and assume that

$$a_1 = \langle Ay, y\rangle = r(A_{11}) > \gamma.$$

Decompose $\mathcal{H}$ as $\mathcal{H}_1 \oplus \mathcal{H}_1^\perp$. Choosing a suitable orthonormal basis for $\mathcal{H}_1$, we may assume that A has operator matrix $\begin{bmatrix} A_{11} & * \\ * & * \end{bmatrix}$, where

$$A_{11} = \operatorname{diag}(a_1, \dots, a_{2k-1}) + \mathrm{i}G$$

with

$$a_1 = r(A_{11}) > \gamma, \qquad a_1 \geq \cdots \geq a_{2k-1} \text{ real}, \qquad G = G^*.$$

Since $a_1 = r(A_{11})$, the $(1, 1)$ entry of G is zero. Now set

$$B_{11} = [\operatorname{diag}(a_1 - a_1, \dots, a_1 - a_k) - iG_1] \oplus 0_{k-1},$$

where G_1 is the leading $k \times k$ principal submatrix of G. Then $a_1 I_k$ is the leading $k \times k$ principal submatrix of $A_{11} + B_{11}$. The principal submatrix of B_{11} obtained by deleting rows and columns with indices $2, \dots, k$ is 0_k. So, letting $B = B_{11} \oplus 0_{\mathcal{H}_1^\perp}$, we see by Lemma 4.1 that $\Lambda_k(B)$ is the singleton $\{0\}$. Thus,

$$r_k(A + B) \geq r_k(A_{11} + B_{11}) \geq a_1 > \gamma = r_k(A) + r_k(B),$$

a contradiction with (4.1). So, we conclude that $|\langle Ax, x\rangle| \leq \gamma$ for every unit vector $x \in \mathcal{H}$.

Now, assume that there is a unit vector $x \in \mathcal{H}$ such that

$$|\langle Ax, x\rangle| = \gamma - \delta \quad \text{ for some } \delta > 0. \tag{4.4}$$

Let $X : \mathbb{C}^k \to \mathcal{H}$ be such that $X^*X = I_k$ and x lies in the range space of X. If $B = XX^* \in \mathcal{S}(\mathcal{H})$, then B is a rank k orthogonal projection. So, $\Lambda_k(B) = [0, 1]$, or $\Lambda_k(B) = \{1\}$ if $\dim \mathcal{H} = 2k - 1$, and $r_k(B) = 1$. We may invoke (4.1) and conclude that there is a sequence $\{\widetilde{\xi}_m\}_{m=1}^\infty$ with $|\widetilde{\xi}_m| = 1$ satisfying

$$\gamma + 1 - 1/m = r_k(A) + r_k(B) - 1/m \leq r_k(\widetilde{\xi}_m A + B) \leq r_k(A) + r_k(B) = \gamma + 1.$$

There exist a sequence of scalars ζ_m with $|\zeta_m| = 1$ and a sequence of linear maps $Y_m : \mathbb{C}^k \to \mathcal{H}$ such that $Y_m^* Y_m = I_k$ and

$$\zeta_m Y_m^*(\widetilde{\xi}_m A + B)Y_m = (\gamma_m + 1)I, \tag{4.5}$$

and

$$\{\gamma_m + 1\} \quad \text{converges to} \quad \gamma + 1. \tag{4.6}$$

Note that $|\langle Bv, v\rangle| \leq \|B\| = 1$ for every unit vector $v \in \mathcal{H}$, and we have shown that $|\langle Au, u\rangle| \leq \gamma$ for every unit vector $u \in \mathcal{H}$. Let $d_j(C)$ denote the (j,j) entry of $C \in M_k$ for $j \in \{1, \ldots, k\}$. It follows from (4.5) that

$$(1) \;\; \{d_j(\zeta_m Y_m^* B Y_m)\} \to 1, \qquad \text{and} \qquad (2) \;\; \{d_j(\zeta_m \widetilde{\xi}_m Y_m^* A Y_m)\} \to \gamma$$

as $m \to \infty$, for every $j \in \{1, \ldots, k\}$. Since $\|\zeta_m Y_m^* B Y_m\| \leq 1$, we have

$$\sum_{j=1}^{k} |d_j(Y_m^* B Y_m)|^2 \leq \operatorname{tr}(Y_m^* B Y_m)^2 \leq k.$$

By (1), $\{\sum_{j=1}^k |d_j(Y_m^* B Y_m)|^2\} \to k$. So

$$\left\{ \operatorname{tr}(Y_m^* B Y_m)^2 - \sum_{j=1}^{k} |d_j(Y_m^* B Y_m)|^2 \right\} \quad \to \quad 0,$$

and hence $\{\zeta_m Y_m^* B Y_m\}$ converges to a diagonal matrix with all diagonal entries approaching 1. Equivalently, $\{\zeta_m Y_m^* B Y_m\} \to I_k$. Since $Y_m^* B Y_m$ is positive semidefinite with eigenvalues in $[0,1]$, we conclude that

$$\{\zeta_m\} \to 1 \qquad \text{and} \qquad \{Y_m^* B Y_m\} \to I_k. \tag{4.7}$$

Since $\{\zeta_m\} \to 1$, we have

$$\{Y_m^*(\widetilde{\xi}_m A + B) Y_m\} \to (\gamma + 1) I.$$

Since $\{Y_m^* B Y_m\} \to I_k$, we have in view of (4.7)

$$\{\widetilde{\xi}_m Y_m^* A Y_m\} = \{\zeta_m^{-1}(\gamma_m + 1) I_k - Y_m^* B Y_m\} \to (\gamma + 1) I_k - I_k = \gamma I_k.$$

Let $Z_m = X^* Y_m \in M_k$ for $m = 1, 2, \ldots$. Then

$$\{Z_m^* Z_m\} = \{Y_m^* X X^* Y_m\} = \{Y_m^* B Y_m\} \to I_k.$$

Passing to subsequences if needed, we may assume that $\{Z_m\}$ converges to a matrix $U \in M_k$. Since $\{Z_m^* Z_m\} \to I_k$, we see that U is unitary. Replacing Y_m by $Y_m U^*$, we may assume that $\{Z_m\} \to I_k$. Let $R_m : \mathbb{C}^{q_m} \to \mathcal{H}$ be such that

$$[X|R_m]^*[X|R_m] = I_{k+q_m},$$

and that $[X|R_m]$ and $[X|Y_m]$ have the same range space. Since $Y_m^* Y_m = I_k$, there exist $C_m : \mathbb{C}^k \to \mathbb{C}^k$ and $S_m : \mathbb{C}^k \to \mathbb{C}^{q_m}$ satisfying

$$C_m^* C_m + S_m^* S_m = I_k \qquad \text{and} \qquad Y_m = XC_m + R_m S_m.$$

Indeed, C_m and S_m are taken from the equalities

$$[X|R_m]Q_m = Y_m, \quad Q_m = \begin{bmatrix} C_m \\ S_m \end{bmatrix},$$

for some $(k + q_m) \times k$ matrix Q_m. Since

$$\{C_m\} = \{X^*(XC_m + R_m S_m)\} = \{X^* Y_m\} = \{Z_m\} \to I_k,$$

we see that $\{\|S_m\|\} \to 0$. Let $T_m = X(C_m - I_k) + R_m S_m$. Then $Y_m = X + T_m$ and $\{\|T_m\|\} \to 0$. Consequently,

$$\begin{aligned} \zeta_m^{-1}(\gamma_m + 1)I_k &= Y_m^*(\widetilde{\xi}_m A + B)Y_m = (X + T_m)^*(\widetilde{\xi}_m A + B)(X + T_m) \\ &= X^*(\widetilde{\xi}_m A + B)X + L_m = \widetilde{\xi}_m X^* AX + I_k + L_m \end{aligned}$$

so that

$$[\zeta_m^{-1}(\gamma_m + 1) - 1]I_k = \widetilde{\xi}_m X^* AX + L_m, \tag{4.8}$$

where

$$\{L_m\} := \{T_m^*(\widetilde{\xi}_m A + B)T_m - T_m^*(\widetilde{\xi}_m A + B)X - X^*(\widetilde{\xi}_m A + B)T_m\} \to 0.$$

Since x is in the range space of X, there is a unit vector $v \in \mathbb{C}^k$ such that $x = Xv$. By (4.8), we have

$$\zeta_m^{-1}(\gamma_m+1)-1 = v^*([\zeta_m^{-1}(\gamma_m+1)-1]I_k)v = v^* X^* \widetilde{\xi}_m AXv + v^* L_m v = \langle \widetilde{\xi}_m Ax, x\rangle + v^* L_m v.$$

Note that $\{\zeta_m^{-1}(\gamma_m + 1) - 1\} \to \gamma$ by (4.6) and (4.7), whereas $\{|\langle \widetilde{\xi}_m Ax, x\rangle + v^* L_m v|\} \to \gamma - \delta$ by (4.4). We get a contradiction. So our claim is true, and

$$W(A) = \{\langle Ax, x\rangle : x \in \mathcal{H}, \langle x, x\rangle = 1\} \subseteq \{\gamma e^{\mathrm{i}t} : t \in [0, 2\pi)\}.$$

By the convexity of $W(A)$, we see that $W(A) = \{\alpha\}$ is a singleton. So, $W(A - \alpha I) = \{0\}$ and $A = \alpha I$ is a scalar operator.

Case 2 Suppose $\dim \mathcal{H} = n < 2k - 1$. Note that in this case, $\Lambda_k(X)$ is either empty or a singleton, for every $X \in \mathcal{V}$. It is easy to see that the supremum is attained in

(4.1). Moreover, if $\Lambda_k(X) = \{\mu\}$, then $\Lambda_k(\Re e^{\mathrm{i}t} A) = \{\Re e^{\mathrm{i}t}\mu\}$ for every $t \in [0, 2\pi)$. Now, suppose $A \in \mathcal{V}$ satisfies $r_k(A) = \gamma > 0$ and (4.1). Then $\Lambda_k(A) = \{\gamma\}$; otherwise, replace A by $e^{\mathrm{i}t}A$ for a suitable $t \in [0, 2\pi)$. Now, for any $B \in \mathcal{V}$ with $\Lambda_k(B) = \{\beta\}$ such that $\beta \geq 0$, we have $r_k(e^{\mathrm{i}t}A + B) = \gamma + \beta$ for some $t \in [0, 2\pi)$. Hence, $\Lambda_k(A+B) = \{\mu\}$ with $|\mu| = \gamma + \beta$.

We first show that $\Re A = \gamma I$. If it is not true, there is a unitary $U \in M_n$ such that $\Re A = U^*\mathrm{diag}\,(a_1, \ldots, a_n)U$ with $a_{n-k+1} \geq \cdots \geq a_k$ satisfying $a_{n-k+1} > a_k$. [Here we use the assumption that $n < 2k - 1$ so that $n - k + 1 < k$ and the subscripts of a_{n-k+1} and a_k are in the right range; note that we do not assume $a_1 \geq \cdots \geq a_n$.] Then there exist $b_1, \beta, b_2 \in \mathbb{R}$ such that

$$b_1 > \beta > 0 > b_2 \quad \text{and} \quad \{\beta + a_j : n - k + 1 \leq j \leq k\} \subseteq (L, R)$$

with

$$R = \min\{b_1 + a_j : 1 \leq j \leq n - k\} \quad \text{and} \quad L = \max\{b_2 + a_j : k < j \leq n\}.$$

Let

$$B = U^*(b_1 I_{n-k} \oplus \beta I_{2k-n} \oplus b_2 I_{n-k})U.$$

Since $\lambda_{n-k+1}(B) = \lambda_k(B) = \beta$, we have $\Lambda_k(B) = \{\beta\}$. However, $\Lambda_k(A+B) = \emptyset$ because

$$\{\lambda_j(\Re(A + B)) : n - k < j \leq k\} = \{a_j + \beta : n - k < j \leq k\}$$

is not a singleton. This is a contradiction.

Now, if $\Im A \neq 0$, then up to a unitary similarity we may assume that

$$A = \gamma I + \mathrm{i}\mathrm{diag}\,(g_1, \ldots, g_n) \quad \text{with } g_1 \neq 0. \tag{4.9}$$

Let $B = I_k \oplus 0$. By (4.1), there are $t_1, t_2 \in [0, 2\pi)$ and an $n \times k$ matrix X such that

$$X^*X = I_k \quad \text{and} \quad e^{\mathrm{i}t_1}X^*(e^{\mathrm{i}t_2}A + B)X = (\gamma + 1)I. \tag{4.10}$$

Since $\lambda_k(\Re(e^{\mathrm{i}(t_1+t_2)}A)) = \cos(t_1 + t_2)\gamma$ and $\lambda_1(\Re(e^{\mathrm{i}t_1}B)) = \cos t_1$, we see that

$$\begin{aligned}\gamma + 1 &= \lambda_k(\Re(e^{\mathrm{i}t_1}X^*(e^{\mathrm{i}t_2}A + B)X)) \\ &\leq \lambda_k(\Re(e^{\mathrm{i}(t_1+t_2)}A)) + \lambda_1(\Re(e^{\mathrm{i}t_1}B)) \;=\; \cos(t_1 + t_2)\gamma + \cos t_1,\end{aligned}$$

where the inequality follows from the Courant-Fischer variational characterization of eigenvalues of Hermitian matrices. It follows that $\cos(t_1 + t_2) = \cos t_1 = 1$. Hence, (4.10) yields

$$X^*(A + B)X = (\gamma + 1)I_k$$

which in turn implies (using (4.9)) that

$$I_k = X^*BX = X^*(I_k \oplus 0_{n-k})X.$$

So, $X^* = [U^*|0_{k,n-k}]$, where $U \in M_k$ is unitary. But then by (4.9), we see that g_1 is an eigenvalue of $\Im(X^*(A+B)X)$ so that $X^*(A+B)X \neq (\gamma+1)I_k$, which is a contradiction. $\square$

Combining inequalities (4.2) and (4.3) we see that the supremum in (4.1) is attained, i.e., it can be replaced by the maximum.

Lemma 4.3 *Let $A \in \mathcal{V}$, and let $\mu \in \mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ if $\mathcal{V} = \mathcal{S}(\mathcal{H})$ and $\mathbb{F} = \mathbb{C}$ if $\mathcal{V} = \mathcal{B}(\mathcal{H})$. Then $\mu \notin \overline{\Lambda_k(A)}$ if and only if there exists $\xi \in \mathbb{F}$ such that*

$$r_k(A - \xi I) < |\mu - \xi|.$$

Proof. If $\Lambda_k(A) = \emptyset$, the result is trivial, so suppose $\Lambda_k(A) \neq \emptyset$ in the rest of the proof. The "if" part is easy: Let ξ be such that $r_k(A - \xi I) < |\mu - \xi|$. Then

$$\mu - \xi \notin \overline{\Lambda_k(A - \xi I)} = \overline{\Lambda_k(A)} - \xi,$$

so $\mu \notin \overline{\Lambda_k(A)}$. So we focus on the "only if" part.

Consider first the case $\mathcal{V} = \mathcal{S}(\mathcal{H})$. Let $\mu \in \mathbb{R} \setminus \overline{\Lambda_k(A)}$, where $\overline{\Lambda_k(A)} = [L, R]$. If $\mu > R$, then for $\xi = L$ we have

$$r_k(A - \xi I) = R - L < \mu - L = |\mu - \xi|.$$

If $\mu < L$, then for $\xi = R$ we have

$$r_k(A - \xi I) = R - L < R - \mu = |\mu - \xi|.$$

Let $A \in \mathcal{V}$ with $\mathcal{V} = \mathcal{B}(\mathcal{H})$. Suppose $\mu \notin \overline{\Lambda_k(A)} = K$. Then by convexity of $\overline{\Lambda_k(A)}$ there is a closed half space $\{z \in \mathbb{C} : \Re(\nu z) \geq \alpha\}$ containing K but not containing μ; here $\nu \in \mathbb{C}$, $\alpha \in \mathbb{R}$. Thus, $\Re(\nu\mu) \leq \Re(\nu\zeta) - \epsilon$ for all $\zeta \in K$, where $\epsilon > 0$ is independent of ζ. Now there is $M > 0$ such that

$$|\mu - M\nu^*|^2 = |M\nu|^2 - 2M\Re(\nu\mu) + |\mu|^2 \geq |M\nu|^2 - 2M\Re(\nu\zeta) + |\zeta|^2 + \epsilon = |\zeta - M\nu^*|^2 + \epsilon.$$

for all $\zeta \in K$. Let $\xi = M\nu^*$. Hence

$$r_k(A - \xi I) = \sup\{|\zeta - \xi| : \zeta \in \Lambda_k(A)\} < |\mu - \xi|.$$

$\square$

We are now ready to present the

Proof of Theorem 1.3.

First, we show that the map ϕ is bijective. By the assumption on ϕ, it suffices to show that ϕ is injective. Suppose $A \neq 0$. By Lemmas 2.5 and 3.2, there is B such that $r_k(\phi(B)) = r_k(B) \in \{-\infty, 0\}$ such that $r_k(\phi(A) + \phi(B)) = r_k(A + B) > 0$. By the same lemmas again, we conclude that $\phi(A) \neq 0$. The injectivity of ϕ follows.

Next, we show that $\phi(\alpha I) = \nu I$ for some $|\nu| = |\alpha|$. To see this, let $A = \alpha I$, and we may assume $\alpha \neq 0$. For $C \in \mathcal{V}$ with $\Lambda_k(C) \neq \emptyset$ there exists $B \in \mathcal{V}$ such that $\phi(B) = C$. Then also $\Lambda_k(B) \neq \emptyset$, and by Lemma 4.2 we have

$$\begin{aligned} r_k(\phi(A)) + r_k(C) &= r_k(A) + r_k(B) \;=\; \max\{r_k(\xi A + B) \,:\, |\xi| = 1\} \\ &= \max\{r_k(\phi(\xi A + B)) : |\xi| = 1\}. \end{aligned}$$

But
$$\max\{r_k(\phi(\xi A + B)) : |\xi| = 1\} = \max\{r_k(\xi\phi(A) + C) : |\xi| = 1\}.$$

Thus
$$r_k(\phi(A)) + r_k(C) = \max\{\xi\phi(A) + C) : |\xi| = 1\},$$

so $\phi(A)$ is a scalar matrix by Lemma 4.2. Note that $r_k(\phi(A)) = |\alpha|$, thus $\phi(A) = \nu I$ with $|\nu| = |\alpha|$.

By the above discussion, ϕ is invertible and $\phi(I) = \xi I$ for some $|\xi| = 1$. Define a map $\psi : \mathcal{V} \to \mathcal{V}$ such that $\psi(A) = \xi^{-1}\phi(A)$. Then clearly $\psi(I) = I$.

Now, we prove that

$$\overline{\Lambda_k(A)} = \overline{\Lambda_k(\psi(A))} \qquad \forall \;\; A \in \mathcal{V}. \tag{4.11}$$

To this end, let $A \in \mathcal{V} = \mathcal{S}(\mathcal{H})$. We proceed by showing the equivalent statement $\mathbb{R} \setminus \overline{\Lambda_k(A)} = \mathbb{R} \setminus \overline{\Lambda_k(\psi(A))}$. Let $\mu \in \mathbb{R} \setminus \overline{\Lambda_k(A)}$, so $\mu \notin \overline{\Lambda_k(A)}$. So there exists $\xi \in \mathbb{R}$ such that $r_k(A - \xi I) < |\mu - \xi|$. But then

$$r_k(A - \xi I) = r_k(\psi(A - \xi I)) = r_k(\psi(A) - \xi\psi(I)) = r_k(\psi(A) - \xi I).$$

So $r_k(\psi(A) - \xi I) < |\mu - \xi|$ and thus $\mu \notin \overline{\Lambda_k(\psi(A))}$ by Lemma 4.3. Therefore

$$\mathbb{R} \setminus \overline{\Lambda_k(A)} \subseteq \mathbb{R} \setminus \overline{\Lambda_k(\psi(A))}. \tag{4.12}$$

Using the bijectivity of ψ, and applying (4.12) for ψ^{-1} we obtain the reverse inclusion.

Let $A \in \mathcal{V} = \mathcal{B}(\mathcal{H})$. Let $\mu \in \mathbb{C} \setminus \overline{\Lambda_k(A)}$. By Lemma 4.3 there exists $\xi \in \mathbb{C}$ such that $r_k(A - \xi I) < |\mu - \xi|$. But then

$$r_k(A - \xi I) = r_k(\psi(A - \xi I)) = r_k(\psi(A) - \xi\psi(I)) = r_k(\psi(A) - \xi I).$$

So $r_k(\psi(A) - \xi I) < |\mu - \xi|$ and thus $\mu \notin \overline{\Lambda_k(\psi(A))}$ by the same lemma. Therefore $\mathbb{C} \setminus \overline{\Lambda_k(A)} \subseteq \mathbb{C} \setminus \overline{\Lambda_k(\psi(A))}$, and using invertibility of ψ we obtain the reverse inclusion.

Now, (4.11) is proved. By Theorem 1.2, there exists a unitary U such that $\psi(A) = U^*AU$ or $\psi(A) = U^*A^tU$ for all $A \in \mathcal{V}$. It will then follow that $\phi(A) = \xi U^*AU$ for all $A \in \mathcal{V}$ or $\phi(A) = \xi U^*A^tU$ for all $A \in \mathcal{V}$. $\square$

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