MAPPINGS THAT PRESERVE PAIRS OF OPERATORS WITH ZERO TRIPLE JORDAN PRODUCT

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ABSTRACT. Let $\mathbb{F}$ be a field and $n \geq 3$. Suppose $\mathfrak{S}_1, \mathfrak{S}_2 \subseteq M_n(\mathbb{F})$ contain all rank-one idempotents. The structure of surjections $\phi : \mathfrak{S}_1 \to \mathfrak{S}_2$ satisfying $ABA = 0 \iff \phi(A)\phi(B)\phi(A) = 0$ is determined. Similar results are also obtained for (a) subsets of bounded operators acting on a complex or real Banach space $\mathfrak{X}$, (b) the space of Hermitian matrices acting on n-dimensional vectors over a skew-field $\mathbb{D}$, (c) subsets of self-adjoint bounded linear operators acting on an infinite dimensional complex Hilbert space. It is then illustrated that the results can be applied to characterize mappings ϕ on matrices or operators such that

$$F(ABA) = F(\phi(A)\phi(B)\phi(A)) \qquad \text{for all } A, B,$$

for functions F such as the spectral norm, Schatten p-norm, numerical radius and numerical range, etc.

1. INTRODUCTION

Motivated by theory and applications, many authors have studied mappings on matrices or operators leaving invariant certain subsets, functions, and relations; for example, see [4, 10, 12, 14] and their references. For instance, given a set $\mathfrak{S}$ of matrices or operators, one would like to determine the structure of mappings $\phi : \mathfrak{S} \to \mathfrak{S}$ satisfying

$$F(\phi(A)) = F(A) \qquad \text{for all } A \in \mathfrak{S} \tag{1.1}$$

for a given function F such as the norm, rank, spectrum, numerical range, etc. Many interesting results have been obtained under the additional assumption that

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the mappings ϕ are linear, additive, or multiplicative. Also, depending on motivations of the study, one may assume that $\mathfrak{S}$ is a certain subspace of operators, a semi-group of operators (say, of bounded rank), the set of rank-one idempotents, etc.

When $\mathfrak{S}$ is a subset of the algebra $M_n(\mathbb{F})$ of matrices over a field $\mathbb{F}$, the mappings satisfying (1.1) and some mild algebraic condition will have a nice form such as

$$A \mapsto MA^\sigma N \quad \text{or} \quad A \mapsto M(A^\sigma)^{\mathrm{t}} N$$

for some invertible matrices $M, N \in M_n(\mathbb{F})$ and field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$. Here X^σ is obtained from X by applying σ entrywise. In many cases, MN is a scalar matrix and hence ϕ is a multiple of a Jordan isomorphism, which has many nice algebraic and analytic properties, and leave invariant various interesting functions and matrix sets such as the rank, determinant, spectrum, the set of invertible matrices, the set of rank–k matrices, commuting pairs of matrices, etc. Equally interesting is the behavior of such mappings when $\mathfrak{S}$ is a subset of the algebra $\mathcal{B}(\mathfrak{X})$ of bounded linear operators acting on a real or complex Banach space $\mathfrak{X}$. Often, the mappings satisfying (1.1) are bounded linear or conjugate-linear, while their algebraic structure is similar to the case when $\mathfrak{S} \subseteq M_n(\mathbb{F})$.

Recently, many researchers have been attracted to the challenging problem of characterizing mappings on matrices (respectively, on $\mathcal{B}(\mathfrak{X})$) with some simple preserving properties without any algebraic and analytic assumptions *a priori*. Of course, one cannot "over-simplify" the assumption and consider an arbitrary mapping $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ satisfying (1.1). Otherwise, one can partition $M_n(\mathbb{F})$ into subsets of matrices having the same functional value under F, and then define a mapping ϕ sending matrices in each of these subsets back to itself. One would not get any additional structure for such mappings. On the other hand, there are interesting results showing that $\phi : \mathfrak{S} \to \mathfrak{S}$ will have nice structure if

$$F(\phi(A) * \phi(B)) = F(A * B) \qquad \text{for all } A, B \in \mathfrak{S} \tag{1.2}$$

for some suitable operation "$*$" and function F. For example, if $F(A*B) = \|A-B\|$ then ϕ has the form $UAV + \phi(0)$ or $UA^{\mathrm{t}}V + \phi(0)$ for some unitary U and V; if $F(A * B) = \|A + B\|$ then ϕ has the form $A \mapsto UAV$ or $A \mapsto UA^{\mathrm{t}}V$ for some unitary U and V; if $F(A*B) = \|AB\|$ then ϕ has the form $A \mapsto \mu_A U^*AU$ for some unitary U and unimodular scalar μ_A; if ϕ is bijective and $F(A*B) = \mathrm{rank}(A-B)$ then ϕ has the form $A \mapsto MAN + \phi(0)$ or $A \mapsto MA^{\mathrm{t}}N + \phi(0)$ for some invertible M and N in $M_n(\mathbb{F})$, etc.; for example, see [2, 3, 14, 15].

In [3], the authors consider such problems on $M_n(\mathbb{F})$ for the usual product $A*B = AB$. It turns out that it is helpful to establish the basic result concerning the mappings $\phi : M_n(\mathbb{F}) \to M_n(\mathbb{F})$ with the property $AB = 0$ if and only if $\phi(A)\phi(B) = 0$. This may be viewed as the special case of (1.2) when $F : M_n(\mathbb{F}) \to \{0, 1\}$ such that $F(0) = 0$ and $F(X) = 1$ for any nonzero X.

In this paper, we follow this line of investigation and consider the Jordan triple product $A * B = ABA$, and study mappings $\phi : \mathfrak{S} \to \mathfrak{S}$ on subsets of $M_n(\mathbb{F})$ or $\mathcal{B}(\mathfrak{X})$ satisfying (1.2). Again, we obtain the basic result concerning such ϕ that

$$ABA = 0 \quad \text{if and only if} \quad \phi(A)\phi(B)\phi(A) = 0. \tag{1.3}$$

This problem will be treated in Section 2. We will impose very mild assumption on the domain $\mathfrak{S}$, namely, that it contains all rank-one idempotents, so that the results can be applied to various settings. In section 3 we obtain similar results for Hermitian matrices over a skew-field or self-adjoint operators acting on a Hilbert space. Then we apply the results to preserver problems in Section 4.

We always use the following notations in our discussion. Let $\mathbb{F}$ be any (commutative) field and $\mathbb{F}^* := \mathbb{F}\setminus\{0\}$. Denote by $\{\mathbf{e}_1, \ldots, \mathbf{e}_n\}$ the standard basis (of column vectors) for $\mathbb{F}^n$, and denote by $\{E_{11}, E_{12}, \ldots, E_{nn}\}$ the standard basis for $M_n(\mathbb{F})$.

Let $\mathfrak{X}^*$ be the dual of Banach space $\mathfrak{X}$, and let $(\mathbf{x} \otimes f) : \mathbf{z} \mapsto \langle \mathbf{z}, f\rangle\, \mathbf{x}$ be the general rank-one operator (here, $\mathbf{x} \in \mathfrak{X}$, $f \in \mathfrak{X}^*$, and $\langle \mathbf{z}, f\rangle = f(\mathbf{z})$). Let X^* be the adjoint of a bounded operator $X : \mathfrak{X} \to \mathfrak{X}$. This operation is also defined for *conjugate linear*, bounded X (i.e., $X(\lambda\mathbf{x}) = \overline{\lambda}X\mathbf{x}$, where $\overline{\lambda}$ is conjugation of complex number), by $(X^*f) : \mathbf{x} \mapsto \overline{\langle X\mathbf{x}, f\rangle}$.

2. Preservers of zeros of Jordan triple products

In this section, we determine the structure of mappings on subsets of matrices or operators preserving pairs having zero Jordan product. We will state the main results and some remarks first, and present the proofs in several subsections.

Theorem 2.1. *Suppose $n \geq 3$, $\mathbb{F}$ is a field, and $\mathfrak{S}_1, \mathfrak{S}_2 \subseteq M_n(\mathbb{F})$ contain all rank-one idempotents. Let $\phi : \mathfrak{S}_1 \to \mathfrak{S}_2$ be surjective and satisfy*

$$ABA = 0 \iff \phi(A)\phi(B)\phi(A) = 0 \quad \text{for all } A, B \in \mathfrak{S}_1. \tag{2.1}$$

Then, there exist an invertible matrix $T \in M_n(\mathbb{F})$, a field automorphism $\sigma : \mathbb{F} \to \mathbb{F}$, and a scalar function $\alpha : \mathfrak{S}_1 \to \mathbb{F}^*$ such that one of the following holds:

(i) $\phi(A) = \alpha(A) \cdot TA^\sigma T^{-1}$ *for all* $A \in \mathfrak{S}_1$.
(ii) $\phi(A) = \alpha(A) \cdot T\,(A^\sigma)^{\mathrm{t}}\, T^{-1}$ *for all* $A \in \mathfrak{S}_1$.

Moreover, if $\mathfrak{S}_1$ also contains all rank-one matrices, then the surjectivity assumption can be removed; the only difference is that σ in (i)–(ii) is (a possibly nonsurjective) *nonzero homomorphism.*

Theorem 2.2. *Suppose $\mathfrak{X}$ is an* infinite-dimensional *Banach space over $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and $\mathfrak{S}_1, \mathfrak{S}_2 \subseteq \mathcal{B}(\mathfrak{X})$ contain all rank-one idempotents. Let $\phi : \mathfrak{S}_1 \to \mathfrak{S}_2$ be surjective and satisfy*

$$ABA = 0 \iff \phi(A)\phi(B)\phi(A) = 0 \quad \text{for all } A, B \in \mathfrak{S}_1. \tag{2.2}$$

Then there is a scalar function $\alpha : \mathfrak{S}_1 \to \mathbb{F}^*$ such that one of the following holds:

(i) *There is a bounded (conjugate) linear bijection $T : \mathfrak{X} \to \mathfrak{X}$ such that $\phi(A) = \alpha(A) \cdot TAT^{-1}$ for all A in $\mathfrak{S}_1$.*
(ii) *The space $\mathfrak{X}$ is reflexive and there is a bounded (conjugate) linear bijection $T : \mathfrak{X}^* \to \mathfrak{X}$ such that $\phi(A) = \alpha(A) \cdot TA^*T^{-1}$ for all A in $\mathfrak{S}_1$.*

The following two corollaries are immediate.

Corollary 2.3. *Suppose $\mathfrak{S}_1 \subseteq M_n(\mathbb{F})$ satisfies the hypothesis of Theorem 2.1, and a mapping $\phi : \mathfrak{S}_1 \to M_n(\mathbb{F})$ satisfies the defining Eq. (2.1), and* contains all rank-one idempotents in its image. *Then, ϕ satisfies the conclusion of Theorem 2.1.*

Corollary 2.4. *Suppose $\mathfrak{S}_1, \mathfrak{S}_2$ satisfy the hypothesis of Theorem 2.1 or Theorem 2.2. Let $\phi : \mathfrak{S}_1 \to \mathfrak{S}_2$ be surjective and satisfy*

$$\operatorname{rank}(ABA) = \operatorname{rank}(\phi(A)\phi(B)\phi(A)) \quad \text{for all } A, B \in \mathfrak{S}_1.$$

Then, ϕ satisfies the conclusion of Theorem 2.1 or Theorem 2.2. Moreover, if $\mathfrak{S}_1 \subseteq M_n(\mathbb{F})$ contains all rank-one matrices, then the surjectivity assumption can be removed; the only difference is that σ in (i)–(ii) is (a possibly nonsurjective) *nonzero homomorphism.*

Several remarks are in order concerning our main results of this section.

Remark 2.5. Note that function α, homomorphism σ, and the invertible matrix T in the conclusion of Theorem 2.1 must be chosen so that $\alpha(A) \cdot TA^\sigma T^{-1} \in \mathfrak{S}_2$ (respectively, $\alpha(A) \cdot T(A^\sigma)^{\mathrm{t}} T^{-1} \in \mathfrak{S}_2$) whenever $A \in \mathfrak{S}_1$. For most applications (see Section 4) and for many domains $\mathfrak{S}_1$ such as the set of rank-one idempotent matrices, the set of matrices with rank bounded by a positive integer, etc., the choice of α, σ, and T is usually very liberal and easy. A similar comment applies to Theorem 2.2.

Remark 2.6. Evidently, the converses of Theorem 2.1 and Theorem 2.2 are valid with suitable choices of α, σ, and T.

Remark 2.7. We believe that the surjectivity assumption in Theorem 2.1 can be removed without any additional assumption. It would be nice to prove or disprove our conjecture.

Remark 2.8. In the infinite dimensional case, nonsurjective mappings satisfying (2.2) may have more complicated structure. For example, in Hilbert spaces $\mathfrak{X}$, one can define $\phi : B(\mathfrak{X}) \to B(\mathfrak{X}) \oplus B(\mathfrak{X}) \subset B(\mathfrak{X})$ by $A \mapsto A \oplus A^*$. Then ϕ is not surjective and satisfies (2.2), but is not of the form Theorem 2.2 (i) or (ii).

Remark 2.9. A similar mapping $\phi : A \oplus B \mapsto A \oplus B^*$ on $\mathfrak{S}_1 = \mathfrak{S}_2 := B(\mathfrak{X}) \oplus B(\mathfrak{X})$ testifies that the structure of surjections with the property (2.2) can be richer, if $\mathfrak{S}_1, \mathfrak{S}_2$ do not contain all rank-one idempotents.

2.1. **Proof for the set of rank-one idempotents.** In this subsection, we first prove Theorem 2.2 for the special case when $\mathfrak{S}_1 = \mathfrak{S}_2 = \mathcal{I}^1$ is the set of rank-one idempotents. Recall that $\mathbf{x} \otimes f$ is a rank-one idempotent if and only if $\langle \mathbf{x}, f \rangle = 1$. In the matrix case, one can identify the linear functional f with a vector $\mathbf{f}$, and identify the operator $\mathbf{x} \otimes f$ with the rank-one matrix $\mathbf{x}\mathbf{f}^{\mathrm{t}}$. We call two idempotents P, Q *orthogonal* if $PQ = 0 = QP$.

We start by proving the injectivity of ϕ.

Lemma 2.10. *Let $P, Q \in \mathcal{I}^1$. We have $P = Q$ if and only if the following implication holds for every rank-one idempotent R:*

$$RPR = 0 \Longrightarrow RQR = 0.$$

Proof. This is obvious for $P = Q$. If $P := \mathbf{x} \otimes f \neq Q := \mathbf{y} \otimes g$ then either $\mathbf{x}, \mathbf{y}$ are linearly independent, or else f, g are. In the first case, choose nonzero functional h with $\langle \mathbf{x}, h \rangle = 0$, and $\langle \mathbf{y}, h \rangle = 1$, to form a rank-one idempotent $R := \mathbf{y} \otimes h$. Obviously, $RPR = 0$, and $RQR = R \neq 0$. We argue similarly when f, g are independent. □

Corollary 2.11. *The surjection $\phi : \mathcal{I}^1 \to \mathcal{I}^1$ from Theorem 2.2 is injective, hence bijective.*

Proof. Suppose $\phi(P) = \phi(Q)$. Then, $RPR = 0 \Longrightarrow \phi(R)\phi(P)\phi(R) = 0 \Longrightarrow \phi(R)\phi(Q)\phi(R) = 0 \Longrightarrow RQR = 0$. By the previous lemma, $P = Q$. □

It is easy to see that $SQS = 0$ is equivalent to $QSQ = 0$ for $S, Q \in \mathcal{I}^1$. With this in mind, given a nonempty subset Ω of rank-one idempotents, we define:

$$\begin{aligned} \Omega^{\triangleright} &:= \{S \in \mathcal{I}^1 : SQS = 0 \text{ for all } Q \in \Omega\} \\ &= \{S \in \mathcal{I}^1 : QSQ = 0 \text{ for all } Q \in \Omega\}. \end{aligned} \tag{2.3}$$

We next associate with each nonzero vector $\mathbf{x} \in \mathfrak{X}$ the set $L_{\mathbf{x}} := \{\mathbf{x} \otimes f : f \in \mathfrak{X}^*,\ \langle \mathbf{x}, f\rangle = 1\}$ of all rank-one idempotents that project onto $\mathrm{Lin}_{\mathbb{F}}\{\mathbf{x}\}$. Similarly, for each nonzero $f \in \mathfrak{X}^*$, we associate the set $R_f := \{\mathbf{x} \otimes f : \mathbf{x} \in \mathfrak{X},\ \langle \mathbf{x}, f\rangle = 1\}$ of all rank-one idempotents with the kernel $\mathrm{Ker}\, f$. Note that $L_{\alpha\mathbf{x}} = L_{\mathbf{x}}$ for every nonzero α. Note also that if $\mathbf{x}$ and $\mathbf{y}$ are linearly independent, then $L_{\mathbf{x}} \cap L_{\mathbf{y}} = \emptyset$. Lastly, note that $L_{\mathbf{x}} \cap R_f$ is either a singleton $\{\alpha\mathbf{x} \otimes f\}$ if there exists α with $\langle \alpha\mathbf{x}, f\rangle = 1$, or else the intersection is empty.

Following [13], we introduce the relation $|$ among rank-one idempotents with the following rule: $P \mid Q$ if both P, Q are in the same $L_{\mathbf{x}}$ or if they are both in the same R_f. We continue by proving that ϕ preserves the relation $|$.

Lemma 2.12. *Let $P := \mathbf{x} \otimes f$ and $Q := \mathbf{x} \otimes g$ be rank-one idempotents in the same $L_{\mathbf{x}}$. Then, $R \in \big(\{P, Q\}^{\triangleright}\big)^{\triangleright}$ if and only if $R = \mathbf{x} \otimes (\lambda f + (1-\lambda)g)$ for some scalar λ.*

Proof. Suppose $R = \mathbf{z} \otimes h \in \big(\{P, Q\}^{\triangleright}\big)^{\triangleright}$. If $\mathbf{z}$ and $\mathbf{x}$ are linearly independent, there exists a nonzero functional h_1, such that $\langle \mathbf{x}, h_1\rangle = 0$ and $\langle \mathbf{z}, h_1\rangle = 1$. Then, $S := \mathbf{z} \otimes h_1$ is a rank-one idempotent. Obviously, $SPS = 0 = SQS$, so $S \in \{P, Q\}^{\triangleright}$. However, $SRS = S \neq 0$, a contradiction.

By transferring the appropriate scalar to the other side of the tensor product, we may thus assume $\mathbf{z} = \mathbf{x}$. Now, if f, g, h are linearly independent, there exists a vector $\mathbf{z}_1 \in (\mathrm{Ker}\, f \cap \mathrm{Ker}\, g) \backslash \mathrm{Ker}\, h$ such that $\langle \mathbf{z}_1, h\rangle = 1$ (see [11, Lemma 2.4.3]). Again, $S := \mathbf{z}_1 \otimes h \in \{P, Q\}^{\triangleright}$, however $SRS \neq 0$, a contradiction. Hence, $h = \lambda f + \mu g$. Moreover, $\langle \mathbf{z}, h\rangle = 1$ gives $\mu = 1 - \lambda$.

On the other hand, if $S \in \{P, Q\}^{\triangleright}$ then either $SP = 0 = SQ$ or else $PS = 0 = QS$. In either case, $SRS = 0$ for every $R = \mathbf{x} \otimes (\lambda f + (1-\lambda)g)$. □

Lemma 2.13. *Let $P := \mathbf{x} \otimes f$ and $Q := \mathbf{y} \otimes f$ be rank-one idempotents in R_f. Then, $R \in \big(\{P, Q\}^{\triangleright}\big)^{\triangleright}$ if and only if $R = (\lambda\mathbf{x} + (1-\lambda)\mathbf{y}) \otimes f$ for some scalar λ.*

Proof. Similar to that of the previous lemma. □

Lemma 2.14. *Let $P, Q \in \mathcal{I}^1$ be* distinct. *Then, we have $P \mid Q$ if and only if $\#\big(\{P, Q\}^{\triangleright}\big)^{\triangleright} \geq 3$.*

Proof. Assume $P \mid Q$, say, $P, Q \in L_{\mathbf{x}}$. Then, $\big(\{P, Q\}^{\triangleright}\big)^{\triangleright}$ consists of the idempotents of the form $\mathbf{x} \otimes (\lambda f + (1-\lambda)g)$. Since $P \neq Q$, the functionals f, g are independent. Hence, we have as many different idempotents in $\big(\{P, Q\}^{\triangleright}\big)^{\triangleright}$, as there are distinct scalars λ. Thus, $\#\big(\{P, Q\}^{\triangleright}\big)^{\triangleright} = \#\mathbb{F} \geq 3$. Similar arguments apply when $P, Q \in R_f$.

Assume lastly $P \nmid Q$. Then, $P = \mathbf{x} \otimes f$, and $Q = \mathbf{y} \otimes g$, and both, $\mathbf{x}, \mathbf{y}$, as well as f, g are linearly independent. Let $R = \mathbf{z} \otimes h \in \big(\{P,Q\}^{\triangleright}\big)^{\triangleright}$. Now, if $\mathbf{x}, \mathbf{y}, \mathbf{z}$ are linearly independent, there exists a functional h_1, with $\langle \mathbf{x}, h_1\rangle = 0 = \langle \mathbf{y}, h_1\rangle$, and $\langle \mathbf{z}, h_1\rangle = 1$. Clearly then, $S := \mathbf{z} \otimes h_1$ is a rank-one idempotent, in $\{P,Q\}^{\triangleright}$, however, $SRS = S \neq 0$, a contradiction. We deduce that $\mathbf{z} = \lambda\mathbf{x} + \mu\mathbf{y}$, and consequently, $R = (\lambda\mathbf{x} + \mu\mathbf{y}) \otimes h$.

Suppose $\mu \neq 0$. We claim that then $h \in \mathbb{F}^* g$. Namely, as $\mathbf{x}, \mathbf{y}$ are linearly independent, and $\dim \mathfrak{X}^* \geq 3$, there exists a functional h_1, linearly independent of g, such that $\langle \mathbf{x}, h_1\rangle = 0$ and $\langle \mu\mathbf{y}, h_1\rangle = 1$. Now, if $h \notin \mathbb{F}^* g$, we could find $\mathbf{z}_1 \in \operatorname{Ker} g$ such that $\langle \mathbf{z}_1, h_1\rangle = 1$, and $\langle \mathbf{z}_1, h\rangle \neq 0$. Then, $S := \mathbf{z}_1 \otimes h_1$ would be a rank-one idempotent, and clearly, $SP = 0 = QS$, so $S \in \{P,Q\}^{\triangleright}$. However, $SRS = (\langle \lambda\mathbf{x} + \mu\mathbf{y}, h_1\rangle \cdot \langle \mathbf{z}_1, h\rangle)\, S \neq 0$, a contradiction. Indeed: $\mu \neq 0$ implies $h \in \mathbb{F}^* g$.

Similarly, we show that $\lambda \neq 0$ would imply $h \in \mathbb{F}^* f$. However, f, g are linearly independent, so either $\lambda = 0$ or else $\mu = 0$. In the first case, a rank-one idempotent R is a scalar multiple of a rank-one idempotent $\mathbf{y} \otimes g = Q$, i.e., $R = Q$. In the second case, $R = P$. Thus, $\#\big(\{P,Q\}^{\triangleright}\big)^{\triangleright} = \#\{P,Q\} = 2$. □

Corollary 2.15. *The bijection ϕ preservers the relation $|$.*

Proof. It was shown in Corollary 2.11 that ϕ is bijective. By the defining Eq. (2.2), $\phi(\Omega^{\triangleright}) = \phi(\Omega)^{\triangleright}$. The rest follows from Lemma 2.14. □

Proof of Theorem 2.2 when $\mathfrak{S}_1 = \mathfrak{S}_2 = \mathcal{I}^1$. We clearly have $P \mid Q$ if and only if $\phi(P) \mid \phi(Q)$. Using the arguments in [13, Proof of Theorem 2.4, or Theorem 2.3, pp. 13–18], we can then prove that either for each nonzero $\mathbf{x}$ there exists a nonzero vector $\hat{\mathbf{x}}$ with $\phi(L_{\mathbf{x}}) = L_{\hat{\mathbf{x}}}$, or else for each nonzero $\mathbf{x}$ there exists a nonzero functional $\hat{g}$ with $\phi(L_{\mathbf{x}}) = R_{\hat{g}}$.

In the former case, suppose $QP = 0$ for rank-one idempotents Q, P. Choose a vector $\mathbf{x}$ with $P \in L_{\mathbf{x}}$. Then, $QL_{\mathbf{x}} = 0 \Longrightarrow QL_{\mathbf{x}}Q = 0 \Longrightarrow \phi(Q)L_{\hat{\mathbf{x}}}\phi(Q) = 0$. It is impossible to have $L_{\hat{\mathbf{x}}}\phi(Q) = 0$, so $\phi(Q)L_{\hat{\mathbf{x}}} = 0$. Since $\phi(P) \in L_{\hat{\mathbf{x}}}$ we deduce $\phi(Q)\phi(P) = 0$. Consequently, ϕ preserves orthogonality among rank-one idempotents. We use a similar argument in the case when $\phi(L_{\mathbf{x}}) = R_{\hat{g}}$. The same argument apply to ϕ^{-1}; so orthogonality is preserved in both direction. By [13, Theorems 2.3 and 2.4], we get the desired conclusion. □

2.2. **Proof for the general case.** In this subsection, we prove the general case of Theorem 2.2 through a series of lemmas. Throughout, I will denote the identity operator, or identity matrix.

Lemma 2.16. *Let $A, B \in \mathcal{B}(\mathfrak{X})\setminus\{0\}$. The following are equivalent.*

(a) *$B = \alpha A$ for some nonzero scalar α.*
(b) *$PAP = 0 \iff PBP = 0$ for all rank-one idempotents P.*

Proof. The implication **(a)⇒(b)** is obvious.

(b)⇒(a). Assume that B is not a multiple of A. We distinguish three cases. Suppose first that there exists a vector $\mathbf{x}$ such that $\mathbf{x}, A\mathbf{x}, B\mathbf{x}$ are independent. Choose $f \in \mathfrak{X}^*$ with $\langle A\mathbf{x}, f\rangle = 0$, and $\langle \mathbf{x}, f\rangle = 1 = \langle B\mathbf{x}, f\rangle$. Then, $P := \mathbf{x} \otimes f$ is a rank-one idempotent, with $PAP = 0$, while $PBP \neq 0$, a contradiction.

Suppose next $A\mathbf{x}, B\mathbf{x}$ are independent, while $\mathbf{x} = \lambda_x A\mathbf{x} + \mu_x B\mathbf{x}$, with, say $\mu_x \neq 0$. Again, choose $f \in \mathfrak{X}^*$ such that $\langle A\mathbf{x}, f\rangle = 0$, and $\langle B\mathbf{x}, f\rangle = 1/\mu_x$. Again, $P := \mathbf{x} \otimes f$ is idempotent, and we get a contradiction as before.

Suppose lastly that $A\mathbf{x}, B\mathbf{x}$ are always linearly dependent. If $\operatorname{Ker} A \subseteq \operatorname{Ker} B$ then $B = \lambda A$, as desired (see [7, Lemma 2.2.i] or [6, Lemma 2.3.1]). Otherwise, pick a (nonzero) vector $\mathbf{x} \in \operatorname{Ker} A \setminus \operatorname{Ker} B$. Now, regardless of linear independence between $\mathbf{x}, B\mathbf{x}$, we could always choose $f \in \mathfrak{X}^*$ with $\langle B\mathbf{x}, f\rangle \neq 0$, and $\langle \mathbf{x}, f\rangle = 1$. Since $\mathbf{x} \in \operatorname{Ker} A$, we get a contradiction as before. □

Lemma 2.17. *Let $A \in \mathcal{B}(\mathfrak{X})\setminus\{0\}$. Then A is not a scalar operator if and only if $PAP = 0$ for some rank-one idempotent P.*

Proof. We prove only the non-trivial part. Suppose $A \in \mathcal{B}(\mathfrak{X})\setminus\{0\}$ is not a scalar. Since $\dim \mathfrak{X} \geq 3$ there exists a vector $\mathbf{u}$ such that $\mathbf{y} := A\mathbf{u}$ and $\mathbf{u}$ are linearly independent. Pick a functional f such that $\langle \mathbf{u}, f\rangle = 1$ and $\langle \mathbf{y}, f\rangle = 0$. Then $P := \mathbf{u} \otimes f$ is a rank-one idempotent and

$$PAP = (\mathbf{u} \otimes f)A(\mathbf{u} \otimes f) = \langle \mathbf{y}, f\rangle\, P = 0. \qquad \square$$

Lemma 2.18. *The following conditions hold:*

(a) *Assume $0 \in \mathfrak{S}_1$. Then also $0 \in \mathfrak{S}_2$. Moreover, $\phi(X) = 0$ if and only if $X = 0$.*
(b) *Assume $\mathfrak{S}_1$ contains nonzero scalar operators. Then the same holds for $\mathfrak{S}_2$. Moreover, $\phi(X)$ is a nonzero scalar operator if and only if X is a nonzero scalar operator.*

Proof. **(a)** Suppose $X \in \mathfrak{S}_1$ is nonzero. Then $X\mathbf{x} \neq 0$ for some vector $\mathbf{x}$. Pick a functional $f \in \mathfrak{X}^*$ such that $\langle \mathbf{x}, f\rangle = 1$ and $\langle X\mathbf{x}, f\rangle \neq 0$. Then, $A := \mathbf{x} \otimes f \in \mathfrak{S}_1$ is a rank-one idempotent, and $AXA \neq 0$ and hence $\phi(A)\phi(X)\phi(A) \neq 0$, so $\phi(X) \neq 0$. Reversed implications, and surjectivity also give $\phi(0) = 0$. Therefore, $0 \in \mathfrak{S}_2$.

(b) Suppose $\mu I \in \mathfrak{S}_1$. If $\phi(\mu I)$ is not a scalar then, by Lemma 2.17, $P\phi(\mu I)P = 0$ for some rank-one idempotent P. By surjectivity, $P = \phi(Q)$ and $\mu Q^2 = Q(\mu I)Q = 0$. So, also $Q^3 = 0$, while $\phi(Q)^3 = P^3 = P \neq 0$, a contradiction.

Conversely, suppose $\phi(X) = \mu I \neq 0$. By (a), $X \neq 0$. If X is non-scalar then, again by Lemma 2.17, we have $PXP = 0$ and hence $\mu\phi(P)^2 = \phi(P)(\mu I)\phi(P) = 0$. So $\phi(P)^3 = 0$, contradicting $P^3 = P \neq 0$. $\square$

Lemma 2.19. *Suppose $\mathfrak{S} \subseteq \mathfrak{X}$ contains all rank-one idempotents, and suppose $A \in \mathfrak{S}$ is not a scalar operator. Then A is a nonzero multiple of a rank-one idempotent if and only if $A^3 \neq 0$ and there does not exist $N \in \mathfrak{S}$ such that $NAN = 0 \neq ANA$.*

Proof. Suppose $\operatorname{rank} A \geq 2$. Since it is not a scalar, there exists $\mathbf{x}$, which is not an eigenvector of A, and there exist vector $\mathbf{y}$ such that $A\mathbf{x}$ and $A\mathbf{y}$ are linearly independent. Then we can choose a nonzero functional f satisfying $\langle A\mathbf{x}, f\rangle = 0$, $\langle \mathbf{x}, f\rangle = 1$ and $\langle A\mathbf{y}, f\rangle \neq 0$. It follows that $N := \mathbf{x} \otimes f$ is a rank-one idempotent in $\mathfrak{S}$. We have $NAN = \langle A\mathbf{x}, f\rangle\, \mathbf{x} \otimes f = 0$, while indeed

$$ANA\mathbf{y} = \langle A\mathbf{y}, f\rangle\, A\mathbf{x} \neq 0.$$

Conversely, assume $A = \mathbf{x} \otimes f$ with $\langle \mathbf{x}, f\rangle \neq 0$. Then $A^3 \neq 0$. Let $N \in \mathfrak{S}$ be arbitrary. If $NAN = (N\mathbf{x}) \otimes (N^* f) = 0$ we conclude that either $N\mathbf{x} = 0$ or $N^* f = 0$. In any case, $ANA = \langle N\mathbf{x}, f\rangle\, A = 0$. $\square$

Corollary 2.20. *Let $\mathfrak{S}_i^0 := (\mathbb{F}^* \mathcal{I}^1) \cap \mathfrak{S}_i$ be the set of nonzero multiples of rank-one idempotents in $\mathfrak{S}_i$. Then $\phi(\mathfrak{S}_1^0) = \mathfrak{S}_2^0$.*

Lemma 2.21. *There exists a bijection $\psi : \mathbb{F}^* \mathfrak{S}_1 \to \mathbb{F}^* \mathfrak{S}_2$ and a nonzero scalar function $\alpha : \mathbb{F}^* \mathfrak{S}_1 \to \mathbb{F}^*$ such that*

$$\psi(A) = \alpha(A)\, \phi(A) \qquad \text{for all } A \in \mathfrak{S}_1.$$

Moreover, ψ preserves rank-one idempotents in both directions and satisfies

$$ABA = 0 \iff \psi(A)\, \psi(B)\, \psi(A) = 0 \qquad \text{for all } A, B \in \mathbb{F}^* \mathfrak{S}_1.$$

Proof. Let $\mathfrak{S}_i^0$ be as in Corollary 2.20. Suppose $X, Y \in \mathfrak{S}_1$ are nonzero such that $X = \lambda Y$. Then clearly $PXP = 0$ if and only if $PYP = 0$ for all $P \in \mathfrak{S}_1^0$. By Corollary 2.20, $Q\phi(X)Q = 0$ if and only if $Q\phi(Y)Q = 0$ for all $Q \in \mathfrak{S}_2^0$. By Lemma 2.18 (a), $\phi(X)$ and $\phi(Y)$ are nonzero; hence $\phi(X)$ and $\phi(Y)$ are scalar multiples of each other by Lemma 2.16. Using surjectivity, we can apply a similar argument to conclude that if $\phi(X) = \lambda\phi(Y)$ are nonzero then X and Y are scalar multiples of each other.

Let $\mathfrak{S}_i/_\sim$ be the set of equivalence classes of $\mathfrak{S}_i$ under the equivalence $X \sim Y \iff \mathbb{F}^* X = \mathbb{F}^* Y$. Define $\widetilde{\psi} : \mathfrak{S}_1/_\sim \to \mathfrak{S}_2/_\sim$ by $\widetilde{\psi}(\mathbb{F}^* A) := \mathbb{F}^* \phi(A)$. This is well defined and injective by the discussion in the preceding paragraph. The surjectivity of ϕ implies the surjectivity of $\widetilde{\psi}$. In each equivalence class $\mathbb{F}^* X$, fix a representative $\dot{X}$ in such a way that if $\mathbb{F}^*\, X$ contains a rank-one idempotent then let $\dot{X}$ be this idempotent. We now extend $\widetilde{\psi}$ to $\psi : \mathbb{F}^*\, \mathfrak{S}_1 \to \mathbb{F}^*\, \mathfrak{S}_2$ by $\psi\,(A) := \lambda \dot{B}$, where $\lambda \dot{A} = A$, and where $\dot{A}$ and $\dot{B}$ are fixed representatives of $\mathbb{F}^*\, A$ and $\widetilde{\psi}\,(\mathbb{F}^*\, A)$, respectively. It is easy to see that such a ψ is bijective. Moreover, $A \in \mathfrak{S}_1$ implies $\mathbb{F}^*\, \psi\,(A) = \mathbb{F}^*\, \phi\,(A)$, so $\psi\,(A) = \alpha\,(A) \cdot \phi\,(A)$ for some nonzero scalar $\alpha(A)$ (if $A = 0$ we may define $\alpha\,(A)$ arbitrarily, say $\alpha\,(0) = 1$). Since $\alpha\,(A) \neq 0$ we obviously have

$$ABA = 0 \iff \phi\,(A)\,\phi\,(B)\,\phi\,(A) = 0 \iff \psi\,(A)\,\psi\,(B)\,\psi\,(A) = 0,$$

whenever $A, B \in \mathfrak{S}_1$. Consequently, $ABA = 0$ if and only if $\psi(A)\psi(B)\psi(A) = 0$ for any $A, B \in \mathbb{F}^*\, \mathfrak{S}_1$. Lastly, ψ preserves rank-one idempotents in both directions, by Corollary 2.20, and the definition of representatives of equivalence classes. □

Proof of Theorem 2.2. Replace ϕ by ψ from the preceding lemma and, retaining the notation, assume without loss of generality that ϕ is bijective, maps $\mathbb{F}^*\, \mathfrak{S}_1$ onto $\mathbb{F}^*\, \mathfrak{S}_2$, preserves the zeros of Jordan triple product, and preserves rank-one idempotents. We can then apply the result in the special case on the restriction $\phi|_{\mathcal{I}^1}$. Suppose it takes the form (ii). Then, the natural embedding $\kappa : \mathfrak{X} \hookrightarrow \mathfrak{X}^{**}$ is surjective. Now, let P be a rank-one idempotent operator, and let $Q := \phi^{-1}(P) = \kappa^{-1}(T^{-1}PT)^*\kappa$. For every nonzero $A \in \mathfrak{S}_1$ we have

$$\begin{aligned} P\phi(A)P = 0 &\iff QAQ = 0 \iff \kappa^{-1}(T^{-1}PT)^*\kappa \cdot A \cdot \kappa^{-1}(T^{-1}PT)^*\kappa = 0. \\ &\iff T^*P^*(T^{-1})^* \underbrace{\kappa \cdot A \cdot \kappa^{-1}}_{=A^{**}} T^*P^*(T^{-1})^* = 0 \\ &\iff P^*(T^{-1})^*A^{**}T^*P^* = 0 \iff (PTA^*T^{-1}P)^* = 0 \\ &\iff PTA^*T^{-1}P = 0. \end{aligned}$$

By Lemma 2.16 and Lemma 2.18 (a), $\phi(A) = \alpha TA^*T^{-1}$ for some nonzero $\alpha = \alpha(A)$.

Similarly we argue if the restriction takes the form (i). □

Proof of Theorem 2.1. The proof of the first part of Theorem 2.1 can be based on obvious adaptation of the arguments in the proof of Theorem 2.2. However, it does not cover the case $\mathbb{F} = \mathbb{Z}_2$ since Lemma 2.14 requires $\#\mathbb{F} \geq 3$. We have found a new approach, that works for all fields. It is based on a single Lemma 2.22 below. With its help, one easily finds that bijection $\phi|_{\mathcal{I}^1} : \mathcal{I}^1 \to \mathcal{I}^1$ preserves maximal sets of pairwise orthogonal rank-one idempotents, hence also orthogonality on $\mathcal{I}^1$. We can then use [13, Theorem 2.3] instead of [13, Theorem 2.4] in the concluding arguments of subsection 2.1. The proof of general case then follows similar arguments as before. □

Lemma 2.22. *Let $Q_1, \dots, Q_n \in \mathcal{I}^1 \subset M_n(\mathbb{F})$ be n idempotents of rank-one. Then, they are pairwise orthogonal if and only if $Q_iQ_jQ_i = 0$ for $i \neq j$, and there exists* no rank-one idempotent B *with $Q_iBQ_i = 0$ for all $i = 1, 2, \dots, n$.*

Proof. Necessity is clear (use $Q_i = TE_{ii}T^{-1}$, and the trace, to deduce that $Q_iBQ_i = 0$ for all $i = 1, \dots, n$ is impossible).

To prove sufficiency, assume that $Q_iQ_jQ_i = 0$ holds for all $i \neq j$, yet idempotents Q_i are not pairwise orthogonal. Write $Q_i = \mathbf{x}_i\mathbf{f}_i^{\mathrm{t}}$, where $\mathbf{f}_i^{\mathrm{t}}\mathbf{x}_i = 1$. It is easy to see that $Q_iQ_jQ_i = 0$ implies that for any pair (i, j), with $i \neq j$, we have $\mathbf{f}_i^{\mathrm{t}}\mathbf{x}_j = 0$ or $\mathbf{f}_j^{\mathrm{t}}\mathbf{x}_i = 0$ but not necessary both. Actually, by our assumption, Q_i *are not* pairwise orthogonal so there must exist a pair (i, j) such that $\mathbf{f}_i^{\mathrm{t}}\mathbf{x}_j \neq 0$ and $\mathbf{f}_j^{\mathrm{t}}\mathbf{x}_i = 0$. Assume without loss of generality that $i = n$ and $j = n - 1$.

Now, if $\dim \mathrm{Lin}_{\mathbb{F}} \{\mathbf{x}_1, \dots, \mathbf{x}_n\} < n$ then there exists a nonzero $\mathbf{f}$ with $\mathbf{f}^{\mathrm{t}}\mathbf{x}_i = 0$ for all i. Pick any $\mathbf{x}$ with $\mathbf{f}^{\mathrm{t}}\mathbf{x} = 1$ to form a rank-one idempotent $B := \mathbf{x}\mathbf{f}^{\mathrm{t}}$. An easy calculation shows that $Q_iBQ_i = 0$ for all $i = 1, \dots, n$.

Otherwise, $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$ is a basis of $\mathbb{F}^n$. Consider the dual base $\{\mathbf{x}_1^*, \dots, \mathbf{x}_n^*\}$ of $\mathbb{F}^n$ (i.e.: $(\mathbf{x}_j^*)^{\mathrm{t}}\mathbf{x}_i = \delta_{ij}$). Let $\beta := -\left(\mathbf{f}_n^{\mathrm{t}}\mathbf{x}_{n-1}\right)^{-1}$, and define

$$B := (\beta\mathbf{x}_{n-1} + \mathbf{x}_n)\,(\mathbf{x}_n^*)^{\mathrm{t}}\,.$$

Then,

$$Q_iBQ_i = \mathbf{x}_i\mathbf{f}_i^{\mathrm{t}}\,(\beta\mathbf{x}_{n-1} + \mathbf{x}_n)\,(\mathbf{x}_n^*)^{\mathrm{t}}\,\mathbf{x}_i\mathbf{f}_i^{\mathrm{t}} = \mathbf{f}_i^{\mathrm{t}}\,(\beta\mathbf{x}_{n-1} + \mathbf{x}_n\,)\,(\mathbf{x}_n^*)^{\mathrm{t}}\,\mathbf{x}_i\,\left(\mathbf{x}_i\mathbf{f}_i^{\mathrm{t}}\right).$$

Now, if $i \neq n$ then $(\mathbf{x}_n^*)^{\mathrm{t}}\,\mathbf{x}_i = 0$, so $Q_iBQ_i = 0$. On the other hand, if $i = n$ then $\mathbf{f}_n^{\mathrm{t}}\,(\beta\mathbf{x}_{n-1} + \mathbf{x}_n) = \beta\,\mathbf{f}_n^{\mathrm{t}}\mathbf{x}_{n-1} + 1 = 0$ so also $Q_nBQ_n = 0$. □

2.3. **Removal of surjectivity assumption in the matrix case.** In this subsection, we show that the surjectivity assumption in Theorem 2.1 can be removed if $\mathfrak{S}_1$ contains all rank-one matrices. To achieve our goal we need the following terminology: With each nonempty subset $\Omega \subseteq M_n(\mathbb{F})$ we associate (cf. Eq. 2.3) the set

$$\Omega^\diamond := \{B \in M_n(\mathbb{F})\backslash\{0\} : ABA = 0 \text{ for every } A \in \Omega\} \subset M_n(\mathbb{F}).$$

Likewise, with each nonzero matrix $A \in M_n(\mathbb{F})$ we associate the set

$$A^\diamond := \{A\}^\diamond = \{B \in M_n(\mathbb{F})\backslash\{0\} : ABA = 0\} \subset M_n(\mathbb{F}).$$

Note that $0 \notin A^\diamond$. Also, note that $A^\diamond = \emptyset$ whenever A is invertible.

We start with two simple technical lemmas.

Lemma 2.23. *Let $A_1, \dots, A_k$ be linearly independent rank-one matrices. Then, $\{0\} \cup \{A_1, \dots, A_k\}^\diamond$ is an $n^2 - k$ dimensional subspace of $M_n(\mathbb{F})$.*

Proof. Write $A_i := \mathbf{x}_i \mathbf{f}_i^{\mathrm{t}}$. Then, $A_i X A_i = 0$ if and only if $0 = \mathbf{f}_i^{\mathrm{t}} X \mathbf{x}_i = \mathrm{Tr}(A_i X)$, the trace of $A_i X$. Note that $\langle X, Y \rangle := \mathrm{Tr}(XY)$ is a pairing, and since $A_1, \dots, A_k$ are linearly independent, there exist $X_1, \dots, X_k$ with $\langle X_j, A_i \rangle = \delta_{ij}$ (see [1]). Thus, the k functionals $\langle \cdot\,, A_i \rangle$ are linearly independent, and their common zero subspace, which equals $\{0\} \cup \{A_1, \dots, A_k\}^\diamond$, is $n^2 - k$ dimensional. □

Lemma 2.24. *Suppose $\sigma : \mathbb{F} \to \mathbb{F}$ is a nonzero field homomorphism, and let $A, B \in M_n(\mathbb{F})$ be nonzero. Then, the following are equivalent:*

(a) *$B = \lambda A^\sigma$ for some nonzero scalar λ.*
(b) *$N^\sigma A^\sigma N^\sigma = 0 \Longleftrightarrow N^\sigma B N^\sigma = 0$ for every rank-one N.*

If, in addition, $\mathrm{rank}\, A = 1 = \mathrm{rank}\, B$ *then (a) is equivalent to:*

(c) *$P^\sigma A^\sigma P^\sigma = 0 \Longleftrightarrow P^\sigma B P^\sigma = 0$ for every rank-one* idempotent *P.*

Proof. The implications **(a)$\Longrightarrow$(b)** and **(a)$\Longrightarrow$(c)** are obvious.

(b)$\Longrightarrow$(a). Let $\mathbf{x}$ be any vector with the property $\mathbf{x} = \mathbf{x}^\sigma$ (say, $\mathbf{x} = \mathbf{e}_i$), and assume erroneously that $\mathbf{b} := B\mathbf{x}$ and $\mathbf{a}^\sigma := A^\sigma \mathbf{x}$ are linearly independent. Let $\mathbf{f}_1, \dots, \mathbf{f}_{n-1}$ be a basis of $\mathbf{a}^\perp := \{\mathbf{f} \in \mathbb{F}^n : \mathbf{f}^{\mathrm{t}} \mathbf{a} = 0\}$. Since the rank equals the maximal dimension of nonzero minor, $\mathbf{f}_1^\sigma, \dots, \mathbf{f}_{n-1}^\sigma$ are also linearly independent. Hence, they are a basis of $(\mathbf{a}^\sigma)^\perp$. Now, $\mathbf{b}$ is independent of $\mathbf{a}^\sigma$, so $(\mathbf{f}_j^\sigma)^{\mathrm{t}} \mathbf{b} \neq 0$ for at least one j. Then, $N^\sigma := (\mathbf{x} \mathbf{f}_j^{\mathrm{t}})^\sigma$ satisfies $N^\sigma B N^\sigma \neq 0$, while $N^\sigma A^\sigma N^\sigma = 0$, a contradiction.

Now, if $\mathrm{rank}\, A^\sigma \geq 2$ then its two columns, say $A^\sigma \mathbf{e}_1$ and $A^\sigma \mathbf{e}_2$, are linearly independent. By the above, $B\mathbf{e}_1 = \lambda_1 A^\sigma \mathbf{e}_1$, and $B\mathbf{e}_2 = \lambda_2 A^\sigma \mathbf{e}_2$, and $B(\mathbf{e}_1 + \mathbf{e}_2) = \lambda A^\sigma (\mathbf{e}_1 + \mathbf{e}_2)$. Hence $\lambda_1 = \lambda = \lambda_2$. Pick i-th column $A^\sigma \mathbf{e}_i$. Then, at least one pair of $\{A^\sigma(\mathbf{e}_1 + \mathbf{e}_i), A^\sigma \mathbf{e}_1\}$, $\{A^\sigma \mathbf{e}_i, A^\sigma \mathbf{e}_2\}$ is linearly independent, and hence $B\mathbf{e}_i = \lambda A^\sigma \mathbf{e}_i$, as well. Consequently, $B = \lambda A^\sigma \neq 0$. We proceed similarly when $\mathrm{rank}\, B \geq 2$.

Lastly, assume $\mathrm{rank}\, A^\sigma = 1 = \mathrm{rank}\, B$. We prove **(c)$\Longrightarrow$(a)**. Note that $A^\sigma = (\mathbf{x}_0 \mathbf{f}_0^{\mathrm{t}})^\sigma$, and $B = \mathbf{y}_0 \mathbf{g}_0^{\mathrm{t}}$. Fix any nonzero $\mathbf{z} \in \mathbf{f}_0^\perp$. We can find n linearly independent $\mathbf{h}_i$ such that $P_i := \mathbf{z} \mathbf{h}_i^{\mathrm{t}}$ are rank-one idempotents. Obviously, $(P_i A P_i)^\sigma = 0$, so also $0 = P_i^\sigma B P_i^\sigma = \big((\mathbf{h}_i^\sigma)^{\mathrm{t}} \mathbf{y}_0\big) \cdot (\mathbf{g}_0^{\mathrm{t}} \mathbf{z}^\sigma) \cdot P_i^\sigma$. Since $\mathbf{h}_1^\sigma, \dots, \mathbf{h}_n^\sigma$ are also independent, hence a basis of $\mathbb{F}^n$, $\big((\mathbf{h}_i^\sigma)^{\mathrm{t}} \mathbf{y}_0\big)$ cannot be always zero. Therefore, $\mathbf{g}_0^{\mathrm{t}} \mathbf{z}^\sigma = 0$. Recall that $\mathbf{z} \in \mathbf{f}_0^\perp$ was arbitrary, so this implies $\{0\} = \mathbf{g}_0^{\mathrm{t}} \cdot \mathrm{Lin}_{\mathbb{F}}(\mathbf{f}_0^\perp)^\sigma = \mathbf{g}_0^{\mathrm{t}} \cdot (\mathbf{f}_0^\sigma)^\perp$. Consequently, $\mathbf{g}_0 \in \mathbb{F}\, \mathbf{f}_0^\sigma$. Dual arguments give $\mathbf{y}_0 \in \mathbb{F}\, \mathbf{x}_0^\sigma$, which finally establishes $B \in \mathbb{F} A^\sigma$. □

We continue with the following observation.

Lemma 2.25. *If $\phi(A) = 0$ then $A = 0 \in \mathfrak{S}_1$.*

Proof. Similar to that of Lemma 2.18 (a). $\square$

Lemma 2.26. *Let $\mathfrak{S}$ be any of the subsets $\mathfrak{S}_1, \mathfrak{S}_2$. Suppose $A \in \mathfrak{S}$ be* nonzero. *Then the following are equivalent:*

(a) $\operatorname{rank} A = 1$.
(b) *There exist n^2-1 matrix tuples $(X_1, C_1), \ldots, (X_{n^2-1}, C_{n^2-1}) \in (A^\diamond \cap \mathfrak{S}) \times \mathfrak{S}$ with the property: $C_k X_k C_k \neq 0$, while $C_k X_z C_k = 0$ whenever $z \neq k$.*

Proof. Suppose $\operatorname{rank} A = 1$, and write it as $A = UE_{11}V$ for some invertible U, V. Define the $n^2 - 1$ matrix tuples

$$(X_{ij}, C_{ij}) := \big(V^{-1}E_{ij}U^{-1}\,,\, UE_{ji}V\big); \qquad \text{where} \quad (ij) \in \Xi := \{1, \ldots, n\}^2 \setminus \{(11)\}.$$

Obviously, $(X_{ij}, C_{ij}) \in (A^\diamond \cap \mathfrak{S}) \times \mathfrak{S}$. Moreover, $C_{ij}X_{ij}C_{ij} = UE_{ji}V \neq 0$, and $C_{ij}X_{uv}C_{ij} = 0$ whenever $(uv) \in \Xi$ is distinct from (ij).

Conversely, assume (b) holds. Now, if $\operatorname{rank} A \geq 2$ then $A = UPV$ for some invertible U, V and idempotent $P := \sum_{i=1}^{r} E_{ii}$, ($r := \operatorname{rank} A$). Then,

$$A^\diamond \cap \mathfrak{S} = \left[V^{-1} \begin{pmatrix} \mathbf{0}_{r\times r} & * \\ * & * \end{pmatrix} U^{-1} \right] \cap \mathfrak{S}$$

spans at most $n^2 - r^2$ dimensional subspace of matrices. By hypothesis, $C_k X_k C_k \neq 0$, while $C_k X_z C_k = 0$ for $z \neq k$. This easily implies that $X_1, \ldots, X_{n^2-1}$ are (n^2-1) linearly independent matrices in $A^\diamond \cap \mathfrak{S}$ — a contradiction. $\square$

Corollary 2.27. *The mapping ϕ preserves matrices of rank-one.*

Proof. Suppose $\operatorname{rank} A = 1$. Choose (n^2-1) matrix tuples from Lemma 2.26. Since $C_k X_k C_k \neq 0$, each matrix A, X_k, C_k is nonzero. Same holds of their ϕ–images, by Lemma 2.25. Since ϕ preserves zeros of Jordan triple product in both directions, the (n^2-1) matrix tuples $(\phi(X_k), \phi(C_k))$ are also in $(\phi(A)^\diamond \cap \mathfrak{S}_2) \times \mathfrak{S}_2$, and $\phi(C_k)\phi(X_k)\phi(C_k) \neq 0$, while $\phi(C_k)\phi(X_z)\phi(C_k) = 0$ for $z \neq k$. By Lemma 2.26, $\operatorname{rank} \phi(A) = 1$. $\square$

Corollary 2.28. *The mapping ϕ maps rank-one idempotents to nonzero scalar multiples of rank-one idempotents.*

Proof. If A is a rank-one idempotent then $A^3 \neq 0$, so also $\phi(A)^3 \neq 0$. Hence, $\phi(A)$ cannot be a rank-one nilpotent, hence it is a scalar multiple of a rank-one idempotent. $\square$

Note that the assumptions and the end result will not be affected if we replace ϕ by a mapping $A \mapsto \alpha(A) \cdot \phi(A)$, where $\alpha(A) \in \mathbb{F}^*$. We will do so in the sequel, and will choose a function α in such a way that the redefined ϕ preserves rank-one idempotents. Evidently, the redefined ϕ also preserves rank-one nilpotent matrices.

We can now continue our discussion:

Lemma 2.29. *The restriction $\phi|_{\mathcal{I}^1} : \mathcal{I}^1 \to \mathcal{I}^1$ is injective.*

Sketch of the proof. Suppose P, Q are distinct rank-one idempotents. Then, they are linearly independent. Write them as $P = \mathbf{x}\mathbf{f}^{\mathrm{t}}$ and $Q = \mathbf{y}\mathbf{g}^{\mathrm{t}}$, to find rank-one $X \in \mathfrak{S}_1$ with $PXP = 0$, and $QXQ \neq 0$. Thus, also $\phi(P)\phi(X)\phi(P) = 0$, while $\phi(Q)\phi(X)\phi(Q) \neq 0$. This gives $\phi(P) \neq \phi(Q)$. □

Lemma 2.30. *The mapping ϕ preserves orthogonality among rank-one idempotents.*

Proof. Let P_1, P_2 be orthogonal rank-one idempotents. We may add $(n-2)$ rank-one idempotents, to obtain a maximal set of pairwise orthogonal idempotents. Pick a similarity V with $P_i = VE_{ii}V^{-1}$.

Note that $E_{ij}E_{uv}E_{ij} = 0$ whenever $(uv) \neq (ji)$. In contrast, $E_{ij}E_{ji}E_{ij} \neq 0$. Since ϕ preserves zeros of Jordan triple product in both directions, we deduce the similar identities for rank-one matrices $A_{ij} := \phi(VE_{ij}V^{-1})$:

$$\begin{aligned} &0 = A_{ij}A_{uv}A_{ij}; \text{ whenever } (uv) \neq (ji) \\ &0 \neq A_{ij}A_{ji}A_{ij}. \end{aligned} \tag{2.4}$$

These identities easily imply that the n^2 matrices A_{ij} are linearly independent. Moreover, they also imply that $A_{ij} \in \{A_{11}, \dots, A_{nn}\}^{\diamond} \cap \mathfrak{S}_2$, whenever $i \neq j$. Hence, $\{A_{11}, \dots, A_{nn}\}^{\diamond} \cap \mathfrak{S}_2$ contains $n^2 - n$ linearly independent nilpotent matrices A_{ij}. By Lemma 2.23, their linear span equals $\{0\} \cup \{A_{11}, \dots, A_{nn}\}^{\diamond}$, and nilpotents A_{ij} are the basis.

Then, however, idempotents $A_{ii} = \phi(P_i)$, which also satisfy $A_{ii}A_{jj}A_{ii} = 0$ for $i \neq j$, must indeed be pairwise orthogonal: Namely, otherwise, there would exist a rank-one *idempotent* $B \in \{A_{11}, \dots, A_{nn}\}^{\diamond}$, by Lemma 2.22. However, in that case the subspace $\{0\} \cup \{A_{11}, \dots, A_{nn}\}^{\diamond}$ could not be spanned by nilpotents alone, since the trace function would vanish on it — a contradiction. □

Proof of the last assertion of Theorem 2.1. We already know that the redefined ϕ preserves rank-one idempotents, and their orthogonality (in one direction only), and that $\phi|_{\mathcal{I}^1}:\mathcal{I}^1\to\mathcal{I}^1$ is injective. By [13, Theorem 2.3], $\phi|_{\mathcal{I}^1}:P\mapsto TP^\sigma T^{-1}$ or else $\phi|_{\mathcal{I}^1}:P\mapsto T(P^\sigma)^{\mathrm{t}}T^{-1}$, for some nonzero homomorphism $\sigma:\mathbb{F}\to\mathbb{F}$.

Replace ϕ by $T^{-1}\phi(\cdot)T$ or by $(T^{-1}\phi(\cdot)T)^{\mathrm{t}}$, so that the redefined ϕ satisfies $\phi|_{\mathcal{I}^1}: P\mapsto P^\sigma$. Let N be any rank-one matrix. Then, $P^\sigma\phi(N)P^\sigma=\phi(P)\phi(N)\phi(P)=0\iff PNP=0\iff (PNP)^\sigma=P^\sigma N^\sigma P^\sigma=0$ for every rank-one idempotent P. Hence, by (c) of Lemma 2.24, $\phi(N)=\alpha(N)\cdot N^\sigma$ for every rank-one N. Assume with no loss of generality that $\alpha(N)=1$. We then repeat the process with arbitrary nonzero matrix $A\in\mathfrak{S}_1$, to deduce that $\phi(A)=\alpha(A)\cdot A^\sigma$, as claimed. Lastly, if $0\in\mathfrak{S}_1$ then $E_{ij}\phi(0)E_{ij}=\phi(E_{ij})\phi(0)\phi(E_{ij})=0$, so $\phi(0)=0^\sigma=0$. □

3. Self-adjoint operators and Hermitian/symmetric matrices

In this section, we obtain results analogous to those in the last section for self-adjoint operators acting on a complex Hilbert space $\big(H,\langle\cdot,\cdot\rangle\big)$. Given a continuous linear operator $T:H\to H$, we let T^* be its Hilbert-space adjoint, i.e, $\langle T\mathbf{x},\mathbf{y}\rangle=\langle\mathbf{x},T^*\mathbf{y}\rangle$. If a continuous $T:H\to H$ is conjugate-linear then we define T^* uniquely by $\langle T\mathbf{x},\mathbf{y}\rangle=\langle T^*\mathbf{y},\mathbf{x}\rangle$.

Theorem 3.1. *Suppose H is a complex,* infinite-dimensional *Hilbert space, and $\mathfrak{S}\subset\mathcal{B}(H)$ is a subset of self-adjoint operators that contains all rank-one projections. Then, a* bijective *$\phi:\mathfrak{S}\to\mathfrak{S}$ satisfies*

$$ABA=0\iff\phi(A)\phi(B)\phi(A)=0\quad\textit{for all } A,B\in\mathfrak{S}\tag{3.1}$$

if and only if there exists a bounded (conjugate) linear bijection $T:H\to H$ with $T^*T=I=TT^*$, and a scalar function $\alpha:\mathfrak{S}\to\mathbb{R}^*$ with the following two properties:

(i) *$\phi(A)=\alpha(A)\cdot TAT^*$ whenever $A\in\mathfrak{S}$ or $\phi(A)\in\mathfrak{S}$ have spectral points of different signs.*
(ii) *$\operatorname{Ker}\phi(A)=\operatorname{Ker}TAT^*$ and $\overline{\mathrm{Im}\phi(A)}=\overline{\mathrm{Im}TAT^*}$ for all $A\in\mathfrak{S}$.*

Remark 3.2. In particular, this shows that the restriction of ϕ on positive definite operators *has no structure*, i.e., ϕ can arbitrarily permute them.

In the finite dimensional case, the surjectivity and injectivity assumption can be removed, at the expense of a slightly larger domain.

Theorem 3.3. *Let $n\geq 3$, let $\mathbb{F}=\mathbb{R}$ or $\mathbb{C}$, and let $\mathbb{H}_n$ be the set of $n\times n$ real symmetric matrices or the set of $n\times n$ complex Hermitian matrices, respectively. Suppose further $\mathfrak{S}\subseteq\mathbb{H}_n$ is a subset that contains all Hermitian matrices of rank ≤ 2. Then, a mapping $\phi:\mathfrak{S}\to\mathfrak{S}$ satisfies*

$$ABA=0\iff\phi(A)\phi(B)\phi(A)=0\quad\textit{for all } A,B\in\mathfrak{S}\tag{3.2}$$

if and only if there exist a unitary matrix U*, and a scalar function* $\alpha : \mathfrak{S} \to \mathbb{R}^*$ *with the following two properties:*

(i) $\phi(A) = \alpha(A) \cdot UA^\dagger U^*$ *whenever* $A \in \mathfrak{S}$ *or* $\phi(A) \in \mathfrak{S}$ *have eigenvalues of different signs.*

(ii) $\operatorname{Ker}\phi(A) = \operatorname{Ker} UA^\dagger U^*$ *and* $\operatorname{Im}\phi(A) = \operatorname{Im} UA^\dagger U^*$ *for all* $A \in \mathfrak{S}$*.*

Here, $A^\dagger = A$ *or* $A^\dagger = \overline{A}$*.*

In the finite dimensional case, we can also consider mappings on Hermitian matrices over a skew-fields. Below we collect some basic facts about such matrices. We refer to [15] for additional information.

Let $\mathbb{D}$ be a skew-field of characteristic $\operatorname{char}\mathbb{D} \neq 2$. Given two matrices $A = \sum \alpha_{ij} E_{ij}$ and $B := \sum \beta_{ij} E_{ij}$ in $M_n(\mathbb{D})$, we define $AB := \sum \gamma_{ij} E_{ij}$, where $\gamma_{ij} := \sum_k \alpha_{ik}\beta_{kj}$. Also, we let $\operatorname{rank} A$ be the *column rank*, i.e., the dimension of the subspace in the *right* $\mathbb{D}$–vector space $\mathbb{D}^n$, generated by the columns of a matrix A. It is known that this equals the row rank of A in the *left* vector space ${}^n\mathbb{D}$.

Suppose $^- : \mathbb{D} \to \mathbb{D}$ is a skew-field antiisomorphism of order two. Let $\mathcal{F} := \{\lambda \in \mathbb{D} : \lambda = \bar{\lambda}\}$ be a set of its fixed points. Throughout, we will assume that $\mathcal{F}$ is a field, contained in the center of $\mathbb{D}$. For any matrix $A \in M_n(\mathbb{D})$ we let $A^* := \bar{A}^{\mathrm{t}}$ be the transpose of a matrix, obtained from A by applying antiisomorphism $^-$ entry-wise. Then, $(AB)^* = B^*A^*$. Recall [15] that A is *Hermitian*, if $A = A^*$. The $\mathcal{F}$-space of all Hermitian matrices over $\mathbb{D}$ will be denoted by $\mathbb{H}_n(\mathbb{D})$, or even by $\mathbb{H}_n$.

Since $\operatorname{char}\mathbb{D} \neq 2$, every Hermitian matrix A is cogredient to a diagonal matrix, i.e., there exists invertible $P \in M_n(\mathbb{D})$ such that

$$A = P \operatorname{diag}(\lambda_1, \dots, \lambda_r, 0, \dots, 0)\, P^*; \qquad (r := \operatorname{rank} A),$$

where $\lambda_1, \dots, \lambda_r \in \mathcal{F}^* := \mathcal{F}\backslash\{0\}$. Consequently, each Hermitian matrix A can be written as $A = \sum_{i=1}^r \mathbf{x}_i\mathbf{x}_i^*\,\lambda_i$, where $\lambda_i \in \mathcal{F}^*$, and $\mathbf{x}_i$ are linearly independent n–by–1 matrices (= column vectors) in the *right* $\mathbb{D}$-vector space $\mathbb{D}^n$. Note that when $\mathbb{D}$ is commutative and $^-$ is identity then $\mathbb{H}_n(\mathbb{D})$ equals the space of symmetric matrices.

We have the following analog of Theorem 3.3:

Theorem 3.4. *Let* $n \geq 3$*, let* $\mathbb{D}$ *be a skew-field with* $\operatorname{char}\mathbb{D} \neq 2$*, and let* $^- : \mathbb{D} \to \mathbb{D}$ *be a skew-field anti-isomorphism of order two, such that* $\mathcal{F} := \{\lambda \in \mathbb{D} : \lambda = \bar{\lambda}\}$ *is a field, contained in the center of* $\mathbb{D}$ *(possibly,* $^-$ *is identity when* $\mathbb{D}$ *is commutative). Denote by* $\mathfrak{S} \subseteq \mathbb{H}_n(\mathbb{D})$ *a subset of Hermitian matrices relative to* $^-$*, that contains all Hermitian matrices of rank* ≤ 2*.*
Suppose $\phi : \mathfrak{S} \to \mathfrak{S}$ *is a surjective mapping with the property*

$$ABA = 0 \iff \phi(A)\phi(B)\phi(A) = 0 \quad \textit{for all } A, B \in \mathfrak{S}. \tag{3.3}$$

Then, there exist $P \in M_n(\mathbb{F})$ with $P^*P = \lambda I$ for some $\lambda \in \mathcal{F}^*$, a skew-field automorphism $\sigma : \mathbb{D} \to \mathbb{D}$ that commutes with $^-$, and a scalar function $\alpha : \mathfrak{S} \to \mathcal{F}^*$ such that

$$\phi(A) = PA^\sigma P^* \cdot \alpha(A) \qquad \text{for all rank-one } A = \mathbf{x}\mathbf{x}^*.$$

The proofs of the main theorems will be presented in the next three subsections.

3.1. **The proof of Theorem 3.1.** We divide the proof of Theorem 3.1 in a series of lemmas. Let $\mathbb{R}^- := (-\infty, 0)$, and $\mathbb{R}^+ := (0, \infty)$. In addition, if $\mathbf{x} \in H$ we let $\mathbf{x}^* := \langle \cdot, \mathbf{x} \rangle$, where $\langle \cdot, \cdot \rangle$ is a scalar product on H.

Lemma 3.5. *Let H be a complex Hilbert space, and let $A, B \in \mathcal{B}(H)$ be self-adjoint operators. Assume the spectrum,* $\mathrm{Sp}(A)$, *contains both positive and negative numbers. Then, the following are equivalent:*

(a) *$B = \lambda A$ for some nonzero scalar λ.*
(b) *$\langle A\mathbf{x}, \mathbf{x} \rangle = 0 \iff \langle B\mathbf{x}, \mathbf{x} \rangle = 0$ for all normalized vectors $\mathbf{x} \in H$.*

Proof. We only prove the nontrivial part **(b)** $\Longrightarrow$ **(a)**.

Measurable Calculus gives us the decomposition of I into pairwise orthogonal projections $P_1 := \int_{\mathrm{Sp}(A)} \chi_{\mathbb{R}^+}(\xi)\, dE(\xi)$, $P_2 := \int_{\mathrm{Sp}(A)} \chi_{\mathbb{R}^-}(\xi)\, dE(\xi)$, and $P_3 := \int_{\mathrm{Sp}(A)} \chi_{\{0\}}(\xi)\, dE(\xi)$, where χ_Ω is the characteristic function of Ω. Let $A_i := P_i A P_i$; then $A = A_1 \oplus A_2 \oplus A_3$, with $A_3 = 0$.

By the spectral mapping Theorem [11, p. 167–168], $\mathrm{Sp}(A_1) \subseteq \overline{\mathrm{Sp}(A) \cap \mathbb{R}^+}$, and $\mathrm{Sp}(A_2) \subseteq \overline{\mathrm{Sp}(A) \cap \mathbb{R}^-}$. Actually, the equality holds everywhere, since $\mathrm{Sp}(A)\backslash\{0\} = \mathrm{Sp}(A_1 \oplus A_2 \oplus A_3)\backslash\{0\} = (\mathrm{Sp}(A_1) \cup \mathrm{Sp}(A_2) \cup \mathrm{Sp}(A_3))\backslash\{0\}$.

Now, suppose A has spectral points of different signs. Recall that the numerical range of a self-adjoint operator is a convex hull of its spectrum, so there exist two normalized vectors $\mathbf{e}_0 \in \mathrm{Im}P_1$ and $\mathbf{f}_0 \in \mathrm{Im}P_2$ such that $\gamma_0^2 := \langle A_1\mathbf{e}_0, \mathbf{e}_0 \rangle = \langle A\mathbf{e}_0, \mathbf{e}_0 \rangle > 0$, and $-\delta_0^2 := \langle A_2\mathbf{f}_0, \mathbf{f}_0 \rangle = \langle A\mathbf{f}_0, \mathbf{f}_0 \rangle < 0$. We next fix arbitrary normalized vectors $\mathbf{e} \in \mathrm{Im}P_1$ and $\mathbf{f} \in \mathrm{Im}P_2$. Moreover, we choose $x, y \in \mathbb{C}$; $|x|^2 + |y|^2 = 1$ to form normalized $\mathbf{x} := x\mathbf{e} + y\mathbf{f}$. It is elementary that $\langle A\mathbf{x}, \mathbf{x} \rangle = (\gamma_{\mathbf{e}}|x| - \delta_{\mathbf{f}}|y|)(\gamma_{\mathbf{e}}|x| + \delta_{\mathbf{f}}|y|)$, where $\gamma_{\mathbf{e}}^2 := \langle A\mathbf{e}, \mathbf{e} \rangle \geq 0$, and $-\delta_{\mathbf{f}}^2 := \langle A\mathbf{f}, \mathbf{f} \rangle \leq 0$. Hence, by the assumptions,

$$\begin{aligned} &(\gamma_{\mathbf{e}}|x| - \delta_{\mathbf{f}}|y|)(\gamma_{\mathbf{e}}|x| + \delta_{\mathbf{f}}|y|) = 0 \\ &\qquad \iff 0 = \langle B\mathbf{x}, \mathbf{x} \rangle = |x|^2 \langle B\mathbf{e}, \mathbf{e} \rangle + 2\,\mathrm{Re}(x\overline{y}\, \langle B\mathbf{e}, \mathbf{f} \rangle) + |y|^2 \langle B\mathbf{f}, \mathbf{f} \rangle. \end{aligned} \tag{3.4}$$

We have four cases to consider:

Case 1: $\gamma_{\mathbf{e}} \neq 0 \neq \delta_{\mathbf{f}}$. Here, we evaluate (3.4) at real $x := \pm\delta_{\mathbf{f}}/\sqrt{\gamma_{\mathbf{e}}^2 + \delta_{\mathbf{f}}^2}$ and $y := \gamma_{\mathbf{e}}/\sqrt{\gamma_{\mathbf{e}}^2 + \delta_{\mathbf{f}}^2}$. Comparing the two results gives $\gamma_{\mathbf{e}}^2\, \langle B\mathbf{f}, \mathbf{f} \rangle + \delta_{\mathbf{f}}^2\, \langle B\mathbf{e}, \mathbf{e} \rangle = 0$

and also $\gamma_{\mathbf{e}}\delta_{\mathbf{f}}\,\mathrm{Re}(\langle B\mathbf{e},\mathbf{f}\rangle)=0$. Evaluate next at $x:=\delta_{\mathbf{f}}\sqrt{-1}/\sqrt{\gamma_{\mathbf{e}}^2+\delta_{\mathbf{f}}^2}$ and $y:=\gamma_{\mathbf{e}}/\sqrt{\gamma_{\mathbf{e}}^2+\delta_{\mathbf{f}}^2}$, to get additional equation $\mathrm{Im}(\langle B\mathbf{e},\mathbf{f}\rangle)=0$. Hence, for some $\lambda_{\mathbf{ef}}\in\mathbb{R}$ we get

$$\begin{aligned}\big(\langle B\mathbf{e},\mathbf{e}\rangle,\langle B\mathbf{f},\mathbf{f}\rangle\big)&=\lambda_{\mathbf{ef}}\big(\gamma_{\mathbf{e}}^2,-\delta_{\mathbf{f}}^2\big)=\lambda_{\mathbf{ef}}\,\big(\langle A\mathbf{e},\mathbf{e}\rangle,\langle A\mathbf{f},\mathbf{f}\rangle\big),\quad\text{and}\\ \langle B\mathbf{e},\mathbf{f}\rangle&=0.\end{aligned}\tag{3.5}$$

Case 2: $\gamma_{\mathbf{e}}=0\neq\delta_{\mathbf{f}}$. Evaluate (3.4) at real $(x,y)=(\cos t,\sin t)$. With $t=0$ we get $\langle B\mathbf{e},\mathbf{e}\rangle=0$. Hence, we may rewrite (3.4) into: $\sin t\neq 0\iff(\cos t,\sin t)\notin\big\{\big(2\,\mathrm{Re}\langle B\mathbf{e},\mathbf{f}\rangle,\,\langle B\mathbf{f},\mathbf{f}\rangle\big)\big\}^{\perp}$, the orthogonal complement in $\mathbb{C}^2$. This easily gives $\mathrm{Re}\langle B\mathbf{e},\mathbf{f}\rangle=0$. We repeat the arguments with $(x,y)=(\cos t,\sqrt{-1}\sin t)$ to deduce that $\mathrm{Im}\langle B\mathbf{e},\mathbf{f}\rangle=0$, as well. Hence, (3.5) holds even in Case 2.

Case 3: $\gamma_{\mathbf{e}}\neq 0=\delta_{\mathbf{f}}$ is similar to Case 2.

Case 4: $\gamma_{\mathbf{e}}=0=\delta_{\mathbf{f}}$. Then, the left-hand side of (3.4) vanishes. This easily gives that all coefficients on the right-hand are zero, whence (3.5).

Likewise we show the validity of Eq. (3.4), and then use arguments from cases (2)–(4) to deduce Eq. (3.5), when precisely one of $\mathbf{e}$ or $\mathbf{f}$ is replaced by a normalized $\mathbf{g}\in\mathrm{Im}P_3$ (provided that $P_3\neq 0$). Recall now that $\gamma_0^2:=\langle A\mathbf{e}_0,\mathbf{e}_0\rangle>0$, and $-\delta_0^2:=\langle A\mathbf{f}_0,\mathbf{f}_0\rangle<0$. It is then straightforward that, in (3.5), $\lambda:=\lambda_{\mathbf{ef}}$ does not depend on choosing normalized vectors $\mathbf{e}\oplus\mathbf{f}\oplus\mathbf{g}\in\mathrm{Im}P_1\oplus\mathrm{Im}P_2\oplus\mathrm{Im}P_3=H$. This shows that $\langle(B-\lambda A)\mathbf{x},\mathbf{x}\rangle=0$ for every normalized $\mathbf{x}\in H$. Hence, $B=\lambda A$. □

We next prove the following counterpart to Lemma 2.19:

Lemma 3.6. *A nonzero self-adjoint operator $A\in\mathfrak{S}$ is of rank-one if and only if $\Omega_A:=\{B\in\mathfrak{S}\backslash\{0\}:ABA=0\}$ is nonempty and maximal.*

Here, maximal means: If $\Omega_A\subseteq\Omega_N$ for some $N\in\mathfrak{S}\backslash\{0\}$, then $\Omega_N=\Omega_A$.

Proof. Suppose Ω_A is nonempty, maximal. Obviously then, A is singular, so that $0\in\mathrm{Sp}(A)$. Moreover, $A\neq 0$, so there exists nonzero spectral point $\xi\in\mathrm{Sp}(A)$. Let $\Delta\subset\mathrm{Sp}(A)$ be an open disc, centered at ξ, and separating it from 0. By the Measurable Calculus, the projection

$$P:=\int_{\mathrm{Sp}(A)}\chi_\Delta(\xi)\,dE(\xi)$$

is nontrivial (i.e, $P\neq 0,I$), and satisfies $A=PAP\oplus(I-P)A(I-P)$. Measurable Calculus with bounded function $\xi\mapsto\chi_\Delta(\xi)/\xi$ also gives $\tilde{A}\in\mathcal{B}(H)$ such that $\tilde{A}A=P=A\tilde{A}$. Hence, $\mathrm{Im}P\subseteq\mathrm{Im}A$, and $\mathrm{Ker}\,A\subseteq\mathrm{Ker}\,P$. Now, if $ABA=0$

then $B(\operatorname{Im}A) \subseteq \operatorname{Ker} A$, and so $B(\operatorname{Im}P) \subseteq B(\operatorname{Im}A) \subseteq \operatorname{Ker} A \subseteq \operatorname{Ker} P$, which gives $PBP = 0$. Consequently, $\Omega_A \subseteq \Omega_P$.

If P is not of rank-one, we can decompose it into projections $P = P_1 \oplus P_2 \oplus P'$, with $\operatorname{rank} P_1 = 1 = \operatorname{rank} P_2$. By hypothesis, $P_1, P_2 \in \mathfrak{S}$. Obviously, $P_2 \in \Omega_{P_1} \backslash \Omega_P$. Then, however, $\Omega_A \subseteq \Omega_P \subsetneqq \Omega_{P_1}$, contradicting maximality. Hence, $\operatorname{rank} P = 1$. By maximality again, $\Omega_A \subseteq \Omega_P$ implies $\Omega_A = \Omega_P$. We claim this is possible only when $\operatorname{rank} A = 1$: Actually, $\mathfrak{S}$ contains all projections of the form $B = \mathbf{z} \otimes \mathbf{z}^*$. Moreover, $\mathbf{z} \otimes \mathbf{z}^* \in \Omega_A \iff 0 = A(\mathbf{z} \otimes \mathbf{z}^*)A = (A\mathbf{z}) \otimes (A^*\mathbf{z})^* = (A\mathbf{z}) \otimes (A\mathbf{z})^* \iff \mathbf{z} \in \operatorname{Ker} A$. Since $\Omega_A = \Omega_P$, this gives $\operatorname{Ker} A = \operatorname{Ker} P$, which is a subspace of codimension one in $\mathcal{B}(H)$. Therefore, $\operatorname{rank} A = 1$.

To prove the reversed implication note that $B \in \Omega_{\xi \mathbf{x} \otimes \mathbf{x}^*} \iff \langle B\mathbf{x}, \mathbf{x} \rangle = 0$. Hence, $\mathbf{y} \otimes \mathbf{y}^* \in \Omega_{\xi \mathbf{x} \otimes \mathbf{x}^*}$ for every $\mathbf{y} \in \{\mathbf{x}\}^\perp$, the orthogonal complement of a set $\{\mathbf{x}\}$. Therefore, if $\Omega_{\xi \mathbf{x} \otimes \mathbf{x}^*} \subseteq \Omega_N$ for some $N \in \mathfrak{S} \backslash \{0\}$, then $0 = N(\mathbf{y} \otimes \mathbf{y}^*)N = (N\mathbf{y}) \otimes (N\mathbf{y})^*$ for every $\mathbf{y} \in \{\mathbf{x}\}^\perp$, which implies that $\{\mathbf{x}\}^\perp \subseteq \operatorname{Ker} N$. Thus, $0 \neq \operatorname{rank} N \leq 1$, and actually, $N \in \mathbb{R}\mathbf{x} \otimes \mathbf{x}^*$. Obviously then, $\Omega_N = \Omega_{\xi \mathbf{x} \otimes \mathbf{x}^*}$. □

Lemma 3.7. *Assume* $0 \in \mathfrak{S}$. *Then* $\phi(A) = 0$ *if and only if* $A = 0$.

Proof. Suppose $A \neq 0$, and pick $\mathbf{x}$ with $A\mathbf{x} \neq 0$. Then, $A(\mathbf{x} \otimes \mathbf{x}^*)A = (A\mathbf{x}) \otimes (A\mathbf{x})^* \neq 0$. By the assumptions, also $\phi(A)\phi(\mathbf{x} \otimes \mathbf{x}^*)\phi(A) \neq 0$, so $\phi(A) \neq 0$. Reversed implications, with surjectivity give $\phi(0) = 0$. □

Corollary 3.8. *The bijection* ϕ *preserves the set of rank-one operators in* $\mathfrak{S}$. *Moreover, for each nonzero vector* $\mathbf{x}$ *there exists nonzero* $\mathbf{y}$ *such that* $\phi(\mathfrak{S} \cap \mathbb{R}^* \, \mathbf{x} \otimes \mathbf{x}^*) \subseteq \mathfrak{S} \cap \mathbb{R}^* \, \mathbf{y} \otimes \mathbf{y}^*$.

Proof. By Lemma 3.7, $\phi(X) = 0 \iff X = 0$. Hence, by the bijectivity, $\phi(\Omega_X) = \Omega_{\phi(X)}$. It is easy to see that bijection ϕ preserves maximality among the sets Ω_X. Consequently, by Lemma 3.6, ϕ maps the set of rank-one operators in $\mathfrak{S}$ to itself.

To prove the addendum, start with a normalized vector $\mathbf{x}$, and pick any $\lambda \in \mathbb{R}^*$ such that $\lambda \mathbf{x} \otimes \mathbf{x}^* \in \mathfrak{S}$. We already know that $\phi(\mathbf{x} \otimes \mathbf{x}^*) = \xi \mathbf{y} \otimes \mathbf{y}^*$, and $\phi(\lambda \mathbf{x} \otimes \mathbf{x}^*) = \zeta \mathbf{z} \otimes \mathbf{z}^*$ for some normalized $\mathbf{y}$ and $\mathbf{z}$, respectively. It now suffices to show that $\mathbf{y}, \mathbf{z}$ are linearly dependent. Assume otherwise. Then, we could find a normalized $\mathbf{w}$ such that $\langle \mathbf{y}, \mathbf{w} \rangle = 0$, and $\langle \mathbf{z}, \mathbf{w} \rangle \neq 0$. By bijectivity, $\mathbf{w} \otimes \mathbf{w}^* = \phi(B)$. Then, however, $\phi(B)\phi(\mathbf{x} \otimes \mathbf{x}^*)\phi(B) = 0$, and $\phi(B)\phi(\lambda \mathbf{x} \otimes \mathbf{x}^*)\phi(B) \neq 0$. This gives $B(\mathbf{x} \otimes \mathbf{x}^*)B = 0 \neq \lambda \, B(\mathbf{x} \otimes \mathbf{x}^*)B$, a contradiction. □

Lemma 3.9. *There exists a bounded (conjugate) linear bijection $T : H \to H$, with $TT^* = I = T^*T$, and a scalar function $\alpha : \mathfrak{S} \to \mathbb{R}^*$, such that*

$$\phi(P) = \alpha(P) \cdot TPT^*; \qquad (P = \mathbf{x} \otimes \mathbf{x}^*).$$

Proof. Let

$$\mathcal{P} := \{\langle \mathbf{x} \rangle = \mathbb{C}\,\mathbf{x} : \mathbf{x} \in H \backslash \{0\}\}$$

be a projective space. Hence, by Corollary 3.8, ϕ induces a well-defined mapping $\Upsilon : \mathcal{P} \to \mathcal{P}$, with the property

$$\Upsilon\langle \mathbf{x} \rangle := \langle \mathbf{y} \rangle \quad \text{if} \quad \phi(\mathbf{x} \otimes \mathbf{x}^*) \in \mathbb{R}^*\, \mathbf{y} \otimes \mathbf{y}^*.$$

Pick any normalized vectors $\mathbf{x}_1, \mathbf{x}_2$. Now, the subspaces $\langle \mathbf{x}_1 \rangle, \langle \mathbf{x}_2 \rangle$ are orthogonal if and only if $(\mathbf{x}_1 \otimes \mathbf{x}_1^*)(\mathbf{x}_2 \otimes \mathbf{x}_2^*)(\mathbf{x}_1 \otimes \mathbf{x}_1^*) = 0$. This, in turn, is equivalent to $\phi(\mathbf{x}_1 \otimes \mathbf{x}_1^*)\phi(\mathbf{x}_2 \otimes \mathbf{x}_2^*)\phi(\mathbf{x}_1 \otimes \mathbf{x}_1^*) = 0$, i.e., to $\Upsilon\langle \mathbf{x}_1 \rangle$ being orthogonal to $\Upsilon\langle \mathbf{x}_2 \rangle$. In addition, Υ is bijective — just repeat the above arguments with ϕ^{-1}.

By the classical Wigner unitary-antiunitary theorem (see Faure [5, Cor. 4.5] for a short proof), there exists a (conjugate) linear, bijective isometry $T : H \to H$ with $\Upsilon\langle \mathbf{x} \rangle = \langle T\mathbf{x} \rangle$. This gives $\phi(\mathbf{x} \otimes \mathbf{x}^*) = \alpha \cdot T\mathbf{x} \otimes (T\mathbf{x})^* = \alpha \cdot T(\mathbf{x} \otimes \mathbf{x}^*)T^*$ for some nonzero scalar $\alpha = \alpha(\mathbf{x} \otimes \mathbf{x}^*)$. Obviously, a bijective (conjugate) linear isometry also satisfies $T^*T = I = TT^*$. ☐

Proof of Theorem 3.1. The sufficiency part is easy. Sketch: we assume $(T, \alpha(X)) = (I, 1)\ \forall X$, and let $ABA = 0$. If B has spectral points of different signs then $\phi(B) = B$, hence $\phi(B)(\overline{\mathrm{Im}\phi(A)}) = B\,\overline{\mathrm{Im}A} \subseteq \mathrm{Ker}\,A = \mathrm{Ker}\,\phi(A)$, giving $\phi(A)\phi(B)\phi(A) = 0$. If, on the other hand, $\mathrm{Sp}(B) \subseteq [0, \infty)$ then $ABA = 0$ implies $(\sqrt{B}A)^*(\sqrt{B}A) = 0$, so $BA = 0$, hence, $\overline{\mathrm{Im}\phi(A)} \subseteq \mathrm{Ker}\,\phi(B)$, i.e., $\phi(A)\phi(B)\phi(A) = 0$. Similarly we see that $\phi(A)\phi(B)\phi(A) = 0$ implies $ABA = 0$.

To prove the necessity we assume, with no loss of generality that, in Lemma 3.9, $\alpha(\mathbf{x}\otimes\mathbf{x}^*) = 1$. Also, we may replace ϕ by $T^*\phi(\cdot)T$ to achieve that $\phi(\mathbf{x}\otimes\mathbf{x}^*) = \mathbf{x}\otimes\mathbf{x}^*$.

Choose now any $A \in \mathfrak{S}$ with both positive and negative spectral points. Note that $(\mathbf{x} \otimes \mathbf{x}^*)A(\mathbf{x} \otimes \mathbf{x}^*) = 0 \iff \langle A\mathbf{x}, \mathbf{x} \rangle = 0$. Consequently, by the assumptions, $\langle A\mathbf{x}, \mathbf{x} \rangle = 0 \iff \langle \phi(A)\mathbf{x}, \mathbf{x} \rangle = 0$. By Lemma 3.5, $\phi(A) = \alpha(A) \cdot A$.

Applying the above argument to ϕ^{-1}, we see that if $B = \phi(A)$ has spectral points of different signs, then A has also spectral points of different signs. So, if all nonzero spectral points of A have the same signs, then same holds of $B = \phi(A)$. Since $\langle A\mathbf{x}, \mathbf{x} \rangle = 0$ if and only if $0 = \langle B\mathbf{x}, \mathbf{x} \rangle$ we see that A and $B = \phi(A)$ have the same kernel (use $\sqrt{B}$), and also the same closure of image (use $\overline{\mathrm{Im}X} = (\mathrm{Ker}\,X)^\perp$ for self-adjoint X). ☐

3.2. **The proof of Theorem 3.3.** For the purpose of this section only, we let $\vec{ij}$ be an ordered pair (ij), where emphasizing that $i < j$. Also, we associate (cf. Eq. 2.3) with each nonzero matrix $A \in \mathbb{H}_n(\mathbb{C})$ the set

$$A^\diamond := \{B \in \mathbb{H}_n(\mathbb{C})\backslash\{0\} : ABA = 0\}.$$

We start with the technical lemma, which characterizes rank-one *complex* Hermitian matrices in terms of zeros of Jordan triple product. It is based on the fact that $\mathbb{H}_n(\mathbb{C})$ is a real vector space of dimension n^2. The sole purpose of Hermitian matrices $B_k, C_{\vec{ij}}, \widetilde{D}_{\vec{ij}}$ below is to control the linear independence among the corresponding $X_k, Y_{\vec{ij}}, \widetilde{Y}_{\vec{ij}}$.

Lemma 3.10. *Let* $A \in \mathbb{H}_n(\mathbb{C})$ *be* nonzero*. Then the following are equivalent:*

(a) $\operatorname{rank} A = 1$.

(b) *There exist* $n-1$ *matrix tuples* $(X_2, B_2), \dots, (X_n, B_n) \in (A^\diamond \cap \mathfrak{S}) \times \mathfrak{S}$*, and two sets of* $n(n-1)/2$ *matrix tuples* $(Y_{\vec{ij}}, C_{\vec{ij}})$, $(\widetilde{Y}_{\vec{ij}}, \widetilde{D}_{\vec{ij}}) \in (A^\diamond \cap \mathfrak{S}) \times \mathfrak{S}$*;* $(1 \le i < j \le n)$ *such that*

$$B_k X_k B_k \neq 0 \qquad C_{\vec{ij}} Y_{\vec{ij}} C_{\vec{ij}} \neq 0 \qquad \widetilde{D}_{\vec{ij}} \widetilde{Y}_{\vec{ij}} \widetilde{D}_{\vec{ij}} \neq 0 \qquad (\forall\, k,\ \forall\, \vec{ij}); \tag{3.6}$$

on the one hand, while on the other:

$$B_k X_s B_k = 0 \qquad B_k Y_{\vec{uv}} B_k = 0 \qquad B_k \widetilde{Y}_{\vec{uv}} B_k = 0 \qquad (\forall\, s \neq k,\ \forall\, \vec{uv}) \tag{3.7}$$

$$C_{\vec{ij}} Y_{\vec{uv}} C_{\vec{ij}} = 0 \qquad C_{\vec{ij}} \widetilde{Y}_{\vec{st}} C_{\vec{ij}} = 0 \qquad (\forall\, \vec{uv} \neq \vec{ij},\ \forall\, \vec{st}) \tag{3.8}$$

$$\widetilde{D}_{\vec{ij}} \widetilde{Y}_{\vec{uv}} \widetilde{D}_{\vec{ij}} = 0 \qquad (\forall\, \vec{uv} \neq \vec{ij}). \tag{3.9}$$

Proof. Suppose $\operatorname{rank} A = 1$, and write it as $A = P\lambda E_{11} P^*$ for some invertible P and nonzero scalar λ. Define the $n-1$ matrix tuples

$$(X_k, B_k) := \big((P^{-1})^* E_{kk} P^{-1}\,,\ P E_{kk} P^*\big); \qquad (k = 2, \dots, n),$$

and the first set of $n(n-1)/2$ matrix tuples

$$(Y_{\vec{ij}}, C_{\vec{ij}}) := \Big((P^{-1})^*(E_{ij}+E_{ji})P^{-1},\ P(E_{ii}+E_{ij}+E_{ji}+E_{jj})P^*\Big); \quad (1 \le i < j \le n),$$

and also the second set of $n(n-1)/2$ matrix tuples

$$(\widetilde{Y}_{\vec{ij}}, \widetilde{D}_{\vec{ij}}) := \Big(\sqrt{-1}(P^{-1})^*(E_{ij} - E_{ji})P^{-1},\ P(E_{ij} + E_{ji})P^*\Big); \quad (1 \le i < j \le n).$$

Obviously, $X_k, Y_{\vec{ij}}, \widetilde{Y}_{\vec{ij}} \in A^\diamond \cap \mathfrak{S}$, and $B_k, C_{\vec{ij}}, \widetilde{D}_{\vec{ij}} \in \mathfrak{S}$. Elementary exercise also validates (3.6)–(3.9).

Conversely, assume (b) holds. Now, if $r := \operatorname{rank} A \geq 2$ then $A = PEP^*$ for some invertible P and diagonal $E := \sum_{i=1}^{r} \lambda_i E_{ii}$. Then,

$$A^\diamond \cap \mathfrak{S} = \left[(P^{-1})^* \begin{pmatrix} \mathbf{0}_{r\times r} & * \\ * & * \end{pmatrix} P^{-1} \right] \cap \mathfrak{S}$$

spans at most $n^2 - r^2$ dimensional $\mathbb{R}$-subspace of complex Hermitian matrices.

It is easily seen that the hypothesis of (b) imply that $\{X_j, Y_{\vec{ij}}, \widetilde{Y}_{\vec{ij}} : 1 \leq i < j \leq n\}$ is an $\mathbb{R}$-linearly independent set that consists of $n^2 - 1$ matrices.

(Indeed, assume $\sum_j \alpha_j X_j + \sum \beta_{\vec{uv}} Y_{\vec{uv}} + \sum \gamma_{\vec{uv}} \widetilde{Y}_{\vec{uv}} = 0$. Pre- and post- multiply with B_k. The assumptions (3.6)–(3.7) yield $\alpha_k = 0\ \forall\, k$. Next, pre- and post- multiply with $C_{\vec{ij}}$ to get $\beta_{\vec{ij}} = 0\ \forall\, k$, via (3.6)–(3.8). Pre- and post- multiply with $\widetilde{D}_{\vec{ij}}$ to finish.)

However, the above set of $n^2 - 1$ $\mathbb{R}$-independent matrices lies in $A^\diamond \cap \mathfrak{S}$, a contradiction. □

Remark 3.11. Similar arguments characterize real-symmetric, rank-one matrices: we just omit the third tuple $(\widetilde{Y}_{\vec{ij}}, \widetilde{D}_{\vec{ij}})$ in Lemma 3.10 (b).

Lemma 3.12. *If $\phi(A) = 0$ then also $A = 0$.*

Proof. Similar to the first part of Lemma 3.7. □

Corollary 3.13. *The mapping ϕ preserves Hermitian matrices of rank-one. Moreover, for each nonzero vector $\mathbf{x}$ there exists nonzero $\mathbf{y}$ such that $\phi(\mathbb{R}^*\, \mathbf{x}\mathbf{x}^*) \subseteq \mathbb{R}^*\, \mathbf{y}\mathbf{y}^*$.*

Proof. Suppose $\operatorname{rank} A = 1$. Choose matrix tuples from Lemma 3.10 (b) (see also Remark 3.11 for real symmetric matrices). Identity (3.6) implies that all matrices $A, X_k, B_k, Y_{\vec{ij}}, C_{\vec{ij}}, \widetilde{Y}_{\vec{ij}}, \widetilde{D}_{\vec{ij}}$ are nonzero. Same holds of their ϕ–images, by Lemma 3.12. Since ϕ preserves zeros of Jordan triple product in both directions, the matrix tuples $(\phi(X_k), \phi(B_k))$, $(\phi(Y_{\vec{ij}}), \phi(C_{\vec{ij}}))$, and $(\phi(\widetilde{Y}_{\vec{ij}}), \phi(\widetilde{D}_{\vec{ij}}))$ are also in $(\phi(A)^\diamond \cap \mathfrak{S}) \times \mathfrak{S}$ and satisfy Eqs. (3.6)–(3.9). By Lemma 3.10, $\operatorname{rank} \phi(A) = 1$.

To prove the addendum, start with $\lambda \in \mathbb{R}^*$ and nonzero vector $\mathbf{x}$. Complete it with vectors $\mathbf{x}_2, \dots, \mathbf{x}_n$ to an orthogonal basis of $\mathbb{F}^n$. Then, $P_1 := \mathbf{x}\mathbf{x}^*$ and $P_i := \mathbf{x}_i\mathbf{x}_i^*$ are rank-one matrices, and $P_i P_j P_i \neq 0$ precisely when $i = j$. Same holds of their images $\phi(P_i)$, by the first part and by the defining Eq. (3.2). Hence, $\phi(P_i) = \xi_i \mathbf{y}_i \mathbf{y}_i^* \neq 0$, and vectors $\mathbf{y}_i$ must also be pairwise orthogonal. Now, consider $\phi(\lambda \mathbf{x}\mathbf{x}^*)$. We have $P_2(\lambda \mathbf{x}\mathbf{x}^*)P_2 = 0 = \cdots = P_n(\lambda \mathbf{x}\mathbf{x}^*)P_n$. As before, we deduce $\phi(\lambda \mathbf{x}\mathbf{x}^*) = \xi \mathbf{z}\mathbf{z}^*$, where $\mathbf{z}$ is orthogonal to $\mathbf{y}_2, \dots, \mathbf{y}_2$. This is possible only when $\mathbf{z} \in \mathbb{F}^* \mathbf{y}_1$, so that $\phi(\lambda \mathbf{x}\mathbf{x}^*) \in \mathbb{R}^* \mathbf{y}_1 \mathbf{y}_1^* = \mathbb{R}^*\, \phi(\mathbf{x}\mathbf{x}^*)$, as anticipated. □

Sketch of the proof of Theorem 3.3. We follow the familiar footsteps to prove necessity: By Corollary 3.13, ϕ induces a well-defined mapping $\Upsilon : \mathcal{P}(\mathbb{F}^n) \to \mathcal{P}(\mathbb{F}^n)$ on the projective space, with the property

$$\Upsilon\langle \mathbf{x}\rangle := \langle \mathbf{y}\rangle \quad \text{if} \quad \phi(\mathbf{x}\mathbf{x}^*) \in \mathbb{R}^*\, \mathbf{y}\mathbf{y}^*.$$

To prove that Υ is projective, suppose $\langle \mathbf{x}\rangle \subseteq \langle \mathbf{x}_1\rangle + \langle \mathbf{x}_2\rangle$. Then, $\mathbf{x} = \lambda_1\mathbf{x}_1 + \lambda_2\mathbf{x}_2$. Denote $\langle \mathbf{y}\rangle := \Upsilon\langle \mathbf{x}\rangle$, $\langle \mathbf{y}_1\rangle := \Upsilon\langle \mathbf{x}_1\rangle$, and $\langle \mathbf{y}_2\rangle := \Upsilon\langle \mathbf{x}_2\rangle$.

Now, if $\langle \mathbf{x}_1\rangle = \langle \mathbf{x}_2\rangle$ then $\mathbf{x} \in \mathbb{F}^*\mathbf{x}_1 = \mathbb{F}^*\mathbf{x}_2$, so that $\Upsilon\langle \mathbf{x}\rangle = \Upsilon\langle \mathbf{x}_1\rangle$, by Corollary 3.13. Otherwise, complete $\mathbf{x}_1, \mathbf{x}_2$ with pairwise orthogonal $\mathbf{x}_3, \dots, \mathbf{x}_n \in \{\mathbf{x}_1, \mathbf{x}_2\}^\perp$. Obviously, they are also orthogonal to $\mathbf{x}$. As in the proof of Corollary 3.13 we deduce that $\phi(\mathbf{x}_i\mathbf{x}_i^*) = \xi_i\mathbf{y}_i\mathbf{y}_i^*$, with $\mathbf{y}_3, \dots, \mathbf{y}_n \in \{\mathbf{y}_1, \mathbf{y}_2\}^\perp$ pairwise orthogonal, and orthogonal to $\mathbf{y}$. Therefore, $\mathbf{y} \in \{\mathbf{y}_3, \dots, \mathbf{y}_n\}^\perp = \{\mathbf{y}_1, \mathbf{y}_2\}$, which translates into the desired $\Upsilon\langle \mathbf{x}\rangle \subseteq \Upsilon\langle \mathbf{x}_1\rangle + \Upsilon\langle \mathbf{x}_2\rangle$. As a byproduct: if the subspaces $\langle \mathbf{x}_1\rangle$ and $\langle \mathbf{x}_2\rangle$ are orthogonal then same holds of $\Upsilon\langle \mathbf{x}_1\rangle$ and $\Upsilon\langle \mathbf{x}_2\rangle$.

We may now use the nonsurjective version of Wigner's unitary-antiunitary theorem (see Faure [5, Theorem 4.1]). Consequently, we get a (conjugate) linear isometry $T : \mathbb{F}^n \to \mathbb{F}^n$ such that $\phi(\mathbf{x}\mathbf{x}^*) = \alpha(\mathbf{x}\mathbf{x}^*) \cdot T(\mathbf{x}\mathbf{x}^*)T^*$. In finite-dimensions, T is automatically bijective.

We next follow the proof of Theorem 3.1, just that Measurable Calculus is replaced with unitary diagonalization of complex Hermitian/real-symmetric matrices in Lemma 3.5. As a result: $\phi(A) = \alpha(A) \cdot TAT^*$ holds for every Hermitian matrix in $\mathfrak{S}$, with both positive and negative eigenvalues (and all rank-one Hermitian matrices). This can be easily rewritten into $\phi(A) = \alpha(A) \cdot UAU^*$, or $\phi(A) = \alpha(A) \cdot U\overline{A}U^*$, where U is a unitary matrix.

The final part is different, though, since ϕ^{-1} may not exist: We first replace, if necessary, ϕ by $(1/\alpha(A) \cdot U^*\phi(\cdot)U)^\dagger$ to achieve that the redefined ϕ fixes rank-one matrices in $\mathfrak{S}$. It is easy to see that the set $\{\mathbf{x} \in \mathbb{F}^n \backslash \{0\} : \mathbf{x}^*A\mathbf{x} = 0\} \cup \{0\}$ is *not a vector subspace of* $\mathbb{F}^n$ if and only if the Hermitian matrix A has both positive and negative eigenvalues. Recall $\phi(\mathbf{x}\mathbf{x}^*) = \mathbf{x}\mathbf{x}^*$, so that $\mathbf{x}^*A\mathbf{x} = 0$ if and only if $\mathbf{x}^*\phi(A)\mathbf{x} = 0$. Consequently, if all eigenvalues of A are nonnegative or nonpositive, then same holds of $B = \phi(A)$. As the proof of Theorem 3.1 we see that $\operatorname{Ker} A = \operatorname{Ker} \phi(A)$ and $\operatorname{Im} A = \operatorname{Im} \phi(A)$.

The sufficiency also goes as the proof of Theorem 3.1. $\square$

3.3. **Proof of Theorem 3.4.** Lastly, we prove Theorem 3.4 concerning Hermitian matrices over a skew-field. We proceed in a series of lemmas.

Lemma 3.14. *Assume* $0 \in \mathfrak{S}$. *Then,* $\phi(A) = 0$ *if and only if* $A = 0$.

Proof. Similar to Lemma 3.7. □

To continue, we classify rank–one Hermitian matrices in terms of zeros of the Jordan triple product:

Lemma 3.15. *A nonzero Hermitian* $A \in \mathfrak{S}$ *is a rank–one matrix if and only if* $\Omega_A := \{B \in \mathfrak{S}\backslash\{0\} : ABA = 0\}$ *is nonempty and maximal.*

Here, maximal means: If $\Omega_A \subseteq \Omega_N$ for some $N \in \mathfrak{S}\backslash\{0\}$, then $\Omega_N = \Omega_A$.

Proof. Suppose A is a Hermitian matrix such that Ω_A is nonempty and maximal. Choose invertible $P \in M_n(\mathbb{D})$ with $A = P(\sum_{i=1}^r \lambda_i E_{ii})P^*$, $r := \operatorname{rank} A$. Clearly then, for $N \in \mathfrak{S}\backslash\{0\}$,

$$ANA = 0 \Longleftrightarrow 0 = \Big(\sum_{i=1}^r \lambda_i E_{ii}\Big)P^*NP\Big(\sum_{i=1}^r \lambda_i E_{ii}\Big) \Longleftrightarrow P^*NP = \begin{pmatrix} \mathbf{0}_{r\times r} & * \\ * & * \end{pmatrix}.$$

Consequently, if $ANA = 0$ then so much the more $\tilde{A}N\tilde{A} = 0$, where $\tilde{A} := PE_{11}P^* \in \mathfrak{S}$. This translates into $\Omega_A \subseteq \Omega_{\tilde{A}}$, which, by maximality, further gives $\Omega_A = \Omega_{\tilde{A}}$. We claim this is possible only when $\operatorname{rank} A = 1$: Actually, $\mathfrak{S}$ contains all matrices of the form $B = \mathbf{z}\mathbf{z}^*$. Moreover, $\mathbf{z}\mathbf{z}^* \in \Omega_A \Longleftrightarrow 0 = A\mathbf{z}\mathbf{z}^*A = (A\mathbf{z})(A^*\mathbf{z})^* = (A\mathbf{z})(A\mathbf{z})^* \Longleftrightarrow \mathbf{z} \in \operatorname{Ker} A$. Since $\Omega_A = \Omega_{\tilde{A}}$, this gives $\operatorname{Ker} A = \operatorname{Ker} \tilde{A}$, which is a subspace of codimension one in $\mathbb{D}^n$. Therefore, $\operatorname{rank} A = 1$.

To prove the reversed implication note that $B \in \Omega_{\mathbf{x}\mathbf{x}^*\lambda} \Longleftrightarrow \mathbf{x}^*B\mathbf{x} = 0$. Hence, $B = \mathbf{y}\mathbf{y}^* \in \Omega_{\mathbf{x}\mathbf{x}^*\lambda}$ for every $\mathbf{y} \in \{\mathbf{x}\}^\perp := \{\mathbf{y} \in \mathbb{D}^n : \mathbf{y}^*\mathbf{x} = 0\}$. Therefore, if $\Omega_{\mathbf{x}\mathbf{x}^*\lambda} \subseteq \Omega_N$, then $0 = N(\mathbf{y}\mathbf{y}^*)N = (N\mathbf{y})(N^*\mathbf{y})^* = (N\mathbf{y})(N\mathbf{y})^*$ for every $\mathbf{y} \in \{\mathbf{x}\}^\perp$. This implies $\{\mathbf{x}\}^\perp \subseteq \operatorname{Ker} N$. Thus, $0 \neq \operatorname{rank} N \leq 1$, and actually, $N \in \mathbf{x}\mathbf{x}^*\,\mathcal{F}$. Obviously then, $\Omega_N = \Omega_{\mathbf{x}\mathbf{x}^*\lambda}$. □

Corollary 3.16. *The surjection* ϕ *preserves Hermitian matrices of rank-one.*

Proof. By Lemma 3.14, $\phi(A) = 0 \Longleftrightarrow A = 0$. Hence, $0 \notin \phi(\Omega_X)$. It is now easy to see that a surjection ϕ, which satisfies the defining Eq. (3.3), also satisfies $\phi(\Omega_X) = \Omega_{\phi(X)}$. Moreover, it preserves maximality among the sets Ω_X: this follows at once from $\Omega_{\phi(X)} \subseteq \Omega_{\phi(N)} \Longrightarrow \Omega_X \subseteq \Omega_N$. The implication, on the other hand, must be true; otherwise, there would exist $B \in \mathfrak{S}$, with $XBX = 0 \neq NBN$. Hence, also $\phi(X)\phi(B)\phi(X) = 0 \neq \phi(N)\phi(B)\phi(N)$, which would contradict $\phi(B) \in \Omega_{\phi(X)} \subseteq \Omega_{\phi(N)}$. Lemma 3.15 now finishes the proof. □

Lemma 3.17. *For each nonzero vector* $\mathbf{x}$ *there exists a vector* $\mathbf{y}$ *with the property* $\phi(\mathbf{x}\mathbf{x}^*\mathcal{F}^*) \subseteq \mathbf{y}\mathbf{y}^*\mathcal{F}^*$.

Proof. Let $\lambda, \mu \in \mathcal{F}^*$. By Corollary 3.16, $\operatorname{rank}\phi(\mathbf{x}\mathbf{x}^*\lambda) = 1 = \operatorname{rank}\phi(\mathbf{x}\mathbf{x}^*\mu)$. Consequently, $\phi(\mathbf{x}\mathbf{x}^*\lambda) = \mathbf{y}\mathbf{y}^*\alpha$, and $\phi(\mathbf{x}\mathbf{x}^*\mu) = \mathbf{z}\mathbf{z}^*\beta$ for some $\alpha, \beta \in \mathcal{F}^*$. Plainly, it suffices to prove that $\mathbf{y}$ and $\mathbf{z}$ are $\mathbb{D}$-linearly dependent, since then, $\mathbf{z} = \mathbf{y}\xi$, so that $\mathbf{z}\mathbf{z}^*\beta = \mathbf{y}\xi\bar{\xi}\mathbf{y}^* \cdot \beta = \mathbf{y}\mathbf{y}^*\xi\bar{\xi} \cdot \beta \in \mathbf{y}\mathbf{y}^*\mathcal{F}^*$.

Assume otherwise. Then, we may find a vector $\mathbf{w}$ with $\mathbf{w}^*\mathbf{y} = 0$ and $\mathbf{w}^*\mathbf{z} = 1$. By surjectivity, $\mathbf{w}\mathbf{w}^* = \phi(A)$. Note that $\alpha, \beta \in \mathcal{F}$ are in the center of $\mathbb{D}$, and $\mathbf{y}^*\mathbf{w} = (\mathbf{w}^*\mathbf{y})^* = 0 \in \mathbb{D}$, so

$$(\mathbf{y}\mathbf{y}^*\alpha) \cdot (\mathbf{w}\,\mathbf{w}^*) \cdot (\mathbf{y}\mathbf{y}^*\alpha) = \mathbf{y}(\mathbf{y}^*\mathbf{w})\,(\mathbf{w}^*\mathbf{y}) \cdot \mathbf{y}^*\alpha^2 = 0.$$

In contrast, $(\mathbf{w}^*\mathbf{z})^*(\mathbf{w}^*\mathbf{z}) = \bar{1} \cdot 1 = 1 \in \mathbb{D}$, so

$$(\mathbf{z}\mathbf{z}^*\beta) \cdot (\mathbf{w}\,\mathbf{w}^*) \cdot (\mathbf{z}\mathbf{z}^*\beta) = \mathbf{z}(\mathbf{z}^*\mathbf{w})(\mathbf{w}^*\mathbf{z})\mathbf{z}^*\beta^2 = \mathbf{z}(\mathbf{w}^*\mathbf{z})^*(\mathbf{w}^*\mathbf{z})\mathbf{z}^*\beta^2 = \mathbf{z}\mathbf{z}^*\beta^2 \neq 0.$$

However, the ϕ–pre-images, $(\mathbf{x}\mathbf{x}^*\lambda)A(\mathbf{x}\mathbf{x}^*\lambda)$ and $(\mathbf{x}\mathbf{x}^*\mu)A(\mathbf{x}\mathbf{x}^*\mu)$ are either both zero or both nonzero, since $\lambda, \mu \in \mathcal{F}^*$ are in the center of $\mathbb{D}$. This contradicts (3.3). □

Below we use the idea in [9] to complete our proof.

Proof of the Theorem 3.4. It suffices to show that $\phi(A) \in (PA^\sigma P^*)\,\mathcal{F}^*$ for every rank-one $A = \mathbf{x}\mathbf{x}^*$, where $P \in M_n(\mathbb{D})$ and $\sigma : \mathbb{D} \to \mathbb{D}$ have the stated properties. We proceed in three steps.

Step 1. We claim that

$$\phi(\mathbf{x}\mathbf{x}^*) \in (P\mathbf{x}^\sigma)(P\mathbf{x}^\sigma)^*\,\mathcal{F}^* = \Big(P\big(\mathbf{x}^\sigma(\mathbf{x}^\sigma)^*\big)P^*\Big)\,\mathcal{F}^* \tag{3.10}$$

for some matrix P, and automorphism $\sigma : \mathbb{D} \to \mathbb{D}$. To see this, let

$$\mathcal{P}(\mathbb{D}^n) := \{\langle \mathbf{x} \rangle = \mathbf{x}\mathbb{D} : \mathbf{x} \in \mathbb{D}^n \backslash \{0\}\}$$

be a projective space. Note that $(\mathbf{x}\xi)(\mathbf{x}\xi)^* = \mathbf{x}\mathbf{x}^*\,\xi\bar{\xi} \in \mathbf{x}\mathbf{x}^*\,\mathcal{F}^*$ for $\xi \in \mathbb{D}\backslash\{0\}$. Hence, by Lemma 3.17, ϕ induces a well-defined mapping $\Upsilon : \mathcal{P}(\mathbb{D}^n) \to \mathcal{P}(\mathbb{D}^n)$, with the property

$$\Upsilon\langle \mathbf{x} \rangle := \langle \mathbf{y} \rangle \quad \text{if} \quad \phi(\mathbf{x}\mathbf{x}^*) \in \mathbf{y}\mathbf{y}^*\,\mathcal{F}^*. \tag{3.11}$$

To prove that Υ is projective, suppose $\langle \mathbf{x} \rangle \subseteq \langle \mathbf{x}_1 \rangle + \langle \mathbf{x}_2 \rangle$. Then, $\mathbf{x} = \mathbf{x}_1\xi_1 + \mathbf{x}_2\xi_2$. Denote $\langle \mathbf{y} \rangle := \Upsilon\langle \mathbf{x} \rangle$, $\langle \mathbf{y}_1 \rangle := \Upsilon\langle \mathbf{x}_1 \rangle$, and $\langle \mathbf{y}_2 \rangle := \Upsilon\langle \mathbf{x}_2 \rangle$ and assume erroneously that $\mathbf{y}$ is $\mathbb{D}$-linearly independent of $\mathbf{y}_1, \mathbf{y}_2$. Then, there is $\mathbf{w} \in \mathbb{D}^n$ with $\mathbf{w}^*\mathbf{y} = 1$, while $\mathbf{w}^*\mathbf{y}_1 = 0 = \mathbf{w}^*\mathbf{y}_2$. By surjectivity, $\mathbf{w}\,\mathbf{w}^* = \phi(A)$. Then, $(\mathbf{w}\,\mathbf{w}^*) \cdot (\mathbf{y}_1\mathbf{y}_1^*) \cdot (\mathbf{w}\,\mathbf{w}^*) = 0 = (\mathbf{w}\,\mathbf{w}^*) \cdot (\mathbf{y}_2\mathbf{y}_2^*) \cdot (\mathbf{w}\,\mathbf{w}^*)$, while $(\mathbf{w}\,\mathbf{w}^*) \cdot (\mathbf{y}\mathbf{y}^*) \cdot (\mathbf{w}\,\mathbf{w}^*) = \mathbf{w}\,\mathbf{w}^* \neq 0$.

Same equations hold for ϕ–pre-images, i.e., $A\mathbf{x}_1\mathbf{x}_1^*A = 0 = A\mathbf{x}_2\mathbf{x}_2^*A$, while $A\mathbf{x}\mathbf{x}^*A \neq 0$. However, $A\mathbf{z}\mathbf{z}^*A = A\mathbf{z}(A^*\mathbf{z})^* = A\mathbf{z}(A\mathbf{z})^* = 0$ if and only if $A\mathbf{z} = 0$. Hence, $A\mathbf{x}_1 = 0 = A\mathbf{x}_2$, while $0 \neq A\mathbf{x} = A(\mathbf{x}_1\xi_1 + \mathbf{x}_2\xi_2) = (A\mathbf{x}_1)\xi_1 + (A\mathbf{x}_2)\xi_2 = 0$, a contradiction. It is easy to see that this implies $\langle \mathbf{y}\rangle \subseteq \langle \mathbf{y}_1\rangle + \langle \mathbf{y}_2\rangle$, i.e., $\Upsilon\langle \mathbf{x}\rangle \subseteq \Upsilon\langle \mathbf{x}_1\rangle + \Upsilon\langle \mathbf{x}_2\rangle$, as claimed.

Note that Υ is surjective, since ϕ is. We now apply the (nonsurjective version of) Fundamental Theorem of Projective Geometry [5]. Hence, $\Upsilon\langle \mathbf{x}\rangle = \langle T\mathbf{x}\rangle$ for some σ-semilinear surjection $T : \mathbb{D}^n \to \mathbb{D}^n$. Actually, $\operatorname{Ker} T = 0$, so T is also injective. By (3.11),

$$\phi(\mathbf{x}\mathbf{x}^*) \in (T\mathbf{x})(T\mathbf{x})^*\,\mathcal{F}^*. \tag{3.12}$$

To prove the rest, let $\mathbf{e}_1, \ldots, \mathbf{e}_n$ be a standard basis of right $\mathbb{D}$-vector space $\mathbb{D}^n$, and let P be a matrix with $P\mathbf{e}_i = T\mathbf{e}_i$. Then, $T\mathbf{x} = P\mathbf{x}^\sigma$, and Eq. (3.12) simplifies into $\phi(\mathbf{x}\mathbf{x}^*) \in (P\mathbf{x}^\sigma)(P\mathbf{x}^\sigma)^*\,\mathcal{F}^* = P\big(\mathbf{x}^\sigma(\mathbf{x}^\sigma)^*\big)P^*\,\mathcal{F}^*$, as anticipated in (3.10).

Step 2. We claim that $P^*P = \lambda I$ for some $\lambda \in \mathcal{F}^*$. To see this, recall that $\mathcal{F}$ is a field, contained in the center of $\mathbb{D}$, and that $\big((\mathbf{x}^\sigma)^*D\mathbf{y}^\sigma\big)\cdot\big((\mathbf{x}^\sigma)^*D\mathbf{y}^\sigma\big)^* \in \mathcal{F}$ for any matrix D and vectors $\mathbf{x}, \mathbf{y}$. Consequently, by (3.10):

$$\begin{aligned}\phi(\mathbf{x}\mathbf{x}^*)\phi(\mathbf{y}\mathbf{y}^*)\phi(\mathbf{x}\mathbf{x}^*) &\in \Big(P\big(\mathbf{x}^\sigma(\mathbf{x}^\sigma)^*\big)P^*\cdot P\big(\mathbf{y}^\sigma(\mathbf{y}^\sigma)^*\big)P^*\cdot P\big(\mathbf{x}^\sigma(\mathbf{x}^\sigma)^*\big)P^*\Big)\,\mathcal{F}^*\\ &\subseteq P\mathbf{x}^\sigma(\mathbf{x}^\sigma)^*P^*\cdot\Big(\big((\mathbf{x}^\sigma)^*D\mathbf{y}^\sigma\big)\cdot\big((\mathbf{x}^\sigma)^*D\mathbf{y}^\sigma\big)^*\Big)\,\mathcal{F}^*, \end{aligned}\tag{3.13}$$

where $D := P^*P = D^*$. Put $\mathbf{x} := \mathbf{e}_i$ and $\mathbf{y} := \mathbf{e}_j$. Then, $\mathbf{e}_i^\sigma = \mathbf{e}_i = \overline{\mathbf{e}_i}$, and the same holds for $\mathbf{e}_j$. Moreover, if $i \neq j$ then $\mathbf{x}^*\mathbf{y} = 0$, hence $(\mathbf{x}\mathbf{x}^*)(\mathbf{y}\mathbf{y}^*)(\mathbf{x}\mathbf{x}^*) = 0$, hence the left side of (3.13) is zero, which is possible only if the right side is zero, as well. This gives $\mathbf{e}_i^*D\mathbf{e}_j = 0$, i.e., the off-diagonal entries of D are zero.

Repeat the procedure with $\mathbf{x} := \mathbf{e}_i + \mathbf{e}_j$ and $\mathbf{y} := \mathbf{e}_i - \mathbf{e}_j$ to deduce that all diagonal entries of D are the same, i.e., D is scalar. Actually, $D = D^*$ implies that this scalar is in $\mathcal{F}^*$.

Step 3. It only remains to see that σ commutes with $\bar{\ }$. Put $\mathbf{x} := (\bar{\xi}, 1, 0, \ldots, 0)^*$, and $\mathbf{y} := (1, -\xi, 0, \ldots, 0)^*$ into (3.13). Note that $\mathbf{x}^*\mathbf{y} = \bar{\xi}\cdot 1 + 1\cdot(-\bar{\xi}) = 0$, hence $(\mathbf{x}\mathbf{x}^*)(\mathbf{y}\mathbf{y}^*)(\mathbf{x}\mathbf{x}^*) = 0$, hence the left, and so also the right side of (3.13) are zero. Since D is a scalar, in the center of $\mathbb{D}$, the right side reduces into $0 = (\mathbf{x}^\sigma)^*\mathbf{y}^\sigma = \overline{\xi^\sigma}\cdot 1 - 1\cdot(\bar{\xi})^\sigma$. Indeed: $\overline{\xi^\sigma} = (\bar{\xi})^\sigma$ for every $\xi \in \mathbb{D}$, and Eq. (3.10) further simplifies into $\phi(\mathbf{x}\mathbf{x}^*) \in P\big(\mathbf{x}\mathbf{x}^*\big)^\sigma P^*\,\mathcal{F}^*$, as claimed. □

4. Applications to preservers

In this section, we show that the results in the last two sections can be used to solve many preserver problems efficiently. Throughout this section, $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$. There has been interest in studying preservers of various types of scalar functions on real or complex matrices including:

- the spectral norm $\|A\| = \sup\{(\mathbf{x}^*A^*A\mathbf{x})^{1/2} : \mathbf{x} \in \mathbb{F}^n,\ \mathbf{x}^*\mathbf{x} = 1\}$,
- the Schatten p-norm $S_p(A) = \{\sum_{j=1}^n s_j(A)^p\}^{1/p}$ for any $p \geq 1$, where $s_1(A) \geq \cdots \geq s_n(A)$ are the singular values of A;
- the numerical radius $r(A) = \max\{|\mathbf{x}^*A\mathbf{x}| : \mathbf{x} \in \mathbb{C}^n, \mathbf{x}^*\mathbf{x} = 1\}$.

Using the results in the previous section, we can obtain a general result covering all these cases. In the following, we consider $F : M_n(\mathbb{F}) \to [0, \infty)$, which satisfies some of the following conditions.

(i) $F(A) = 0$ if and only if $A = 0$.

(ii) There is a nonzero $p \in \mathbb{R}$ such that $F(\mu A) = |\mu|^p F(A)$ for all $\mu \in \mathbb{F}$ and $A \in M_n(\mathbb{F})$.

(iii) $F(A) = F(U^*AU)$ for all $U, A \in M_n(\mathbb{F})$ with $U^*U = I_n$.

We have the following result.

Theorem 4.1. *Let $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, $n \geq 3$, and $\mathfrak{S} \subseteq M_n(\mathbb{F})$ contains all rank-one idempotents. Suppose $F : M_n(\mathbb{F}) \to [0, \infty)$ and $\phi : \mathfrak{S} \to \mathfrak{S}$ is surjective and satisfies*

$$F(ABA) = F(\phi(A)\phi(B)\phi(A)) \qquad \text{for all } A, B \in \mathfrak{S}.$$

If F satisfies (i)*, then there exist an invertible $S \in M_n(\mathbb{F})$, a field automorphism σ of $\mathbb{F}$, and $\alpha : \mathfrak{S} \to \mathbb{F}^*$ such that ϕ has the form*

$$A \mapsto \alpha(A) \cdot SA^\sigma S^{-1} \qquad \text{or} \qquad A \mapsto \alpha(A) \cdot S(A^\sigma)^{\mathrm{t}} S^{-1}.$$

If F satisfies (i) – (ii)*, then σ is continuous (i.e., σ is identity or a complex conjugation) in the above conclusion. If F satisfies* (i) – (iii)*, and $\mathfrak{S}$ contains all idempotent and nilpotent matrices of rank-one, then S can be chosen unitary, and $|\alpha(A)| = 1$ for all nonzero $A \in \mathfrak{S}$ in the above conclusion.*

Proof. By Theorem 2.1, if F satisfies (i), then there is an invertible S and a function $\alpha : \mathfrak{S} \to \mathbb{F}^*$ such that ϕ has the form

$$A \mapsto \alpha(A) \cdot SA^\sigma S^{-1} \qquad \text{or} \qquad A \mapsto \alpha(A) \cdot S(A^\sigma)^{\mathrm{t}} S^{-1}. \tag{4.1}$$

Suppose F also satisfies (ii). Then we may replace F by the map $A \mapsto (F(A))^{1/p}$ and assume that $p = 1$. To prove continuity of σ, we consider the restriction of ϕ on rank-one idempotent matrices. If A has rank-one, then A is unitarily similar to A^{t}, and thus $F(A) = F(A^{\mathrm{t}})$. So, we may assume that ϕ satisfies the first form; otherwise, replace ϕ by $A \mapsto \phi(A^{\mathrm{t}})$. Let $A = E_{11} + zE_{12}$, $B = E_{11} + E_{12}$, and $C = E_{21} + E_{22}$. Then $ABA = A$ and $ACA = zA$. Thus,

$$|z| \cdot F(A) = |z| \cdot F(ABA) = |z| \cdot F(\phi(A)\phi(B)\phi(A)) = |z|\,|\alpha(A)\alpha(B)| \cdot F(\phi(A)),$$

which is the same as

$$F(zA) = F(ACA) = F(\phi(A)\phi(C)\phi(A)) = |\sigma(z)|\,|\alpha(A)\alpha(C)| \cdot F(\phi(A)).$$

Putting $z = 1$, we see that $|\alpha(C)| = |\alpha(B)|$. Using this fact, we see that $|\sigma(z)| = |z|$ as asserted.

Now, suppose $\mathfrak{S}$ contains all idempotent and nilpotent matrices of rank-one, and F satisfies (i) – (iii). We first consider the restriction of ϕ on rank-one matrices and prove that a scalar multiple of S is unitary. We will then show that $|\alpha(X)| = 1$ for all $X \in \mathfrak{S}$. As before, we may assume that this restriction has the form $A \mapsto \alpha(A) \cdot SA^\sigma S^{-1}$. Furthermore, if $S = UDV$ is a singular value decomposition, we may replace ϕ by $A \mapsto U^*\phi(\hat{V}^*A\hat{V})U$; ($\hat{V} := V^{\sigma^{-1}}$) and assume that $S = D$ is the diagonal matrix $D = \operatorname{diag}(d_1, \ldots, d_n)$ with $d_1 \geq \cdots \geq d_n > 0$. Then, $\phi(A) = \alpha(A) \cdot DAD^{-1}$ if $A \in \mathfrak{S}$ is a rank-one matrix with integer coefficients. Also, $\phi(E_{ij}) = d_i d_j^{-1}\alpha(E_{ij})E_{ij}$. Therefore,

$$F(E_{jj}) = F(E_{jj}^3) = F(\phi(E_{jj})^3) = |\alpha(E_{jj})|^3 \cdot F(E_{jj})$$

and hence $|\alpha(E_{jj})| = 1$ for all $j = 1, \ldots, n$. Next, observe that

$$F(E_{jj}) = F(E_{jj}(E_{jj} + E_{ji})E_{jj}) = |\alpha(E_{jj})^2\alpha(E_{jj} + E_{ji})| \cdot F(E_{jj}).$$

Consequently, $|\alpha(E_{jj} + E_{ji})| = 1$. Next,

$$F(E_{ij}) = F(E_{ij}E_{ji}E_{ij}) = d_i d_j^{-1}|\alpha(E_{ij})^2\alpha(E_{ji})| \cdot F(E_{ij}),$$

which is the same as

$$F(E_{ij}) = F(E_{ij}(E_{jj} + E_{ji})E_{ij}) = d_i d_j^{-1}|\alpha(E_{ij})^2\alpha(E_{jj} + E_{ji})| \cdot F(E_{ij}).$$

It follows that $|\alpha(E_{ji})| = |\alpha(E_{jj} + E_{ji})| = 1$, whenever $i \neq j$. Hence also $|\alpha(E_{ij})| = 1$, and the last equation gives $d_i d_j^{-1} = 1$. Therefore, $D = \lambda I$ is a scalar, and $S = \lambda UV$. Nothing changes in Eq. (4.1) if we replace S by $\lambda^{-1}S = UV$. Thus, S can be chosen unitary.

For simplicity we may assume $S = I$. Recall that we have already shown $|\alpha(E_{ij})| = 1$ for all i, j. Consider a general $X \in \mathfrak{S}\backslash\{0\}$. Now, if X has the (ij) entry equal to a nonzero number μ then, by the assumption on ϕ, and Eq. (4.1):

$$\begin{aligned}|\mu| \cdot F(E_{ji}) &= F(E_{ji}XE_{ji}) = F\big(\phi(E_{ji})\phi(X)\phi(E_{ji})\big)\\ &= |\alpha(E_{ji})^2\alpha(X)| \cdot F\big((E_{ji}XE_{ji})^\tau\big) = |\alpha(E_{ji})^2\alpha(X)|\,|\sigma(\mu)| \cdot F\big(E_{ji}^\tau\big),\end{aligned}$$

where A^τ denotes A^σ or $(A^\sigma)^{\mathrm{t}}$. Note that $|\mu| = |\sigma(\mu)|$, and $F(E_{ji}^{\mathrm{t}}) = F(E_{ji})$, so that $|\alpha(X)| = 1$. $\square$

Remark 4.2. Note that one needs to assume that $\mathfrak{S}$ contains all rank-one nilpotents to get the last assertion. For example, define $F(X) = |\operatorname{Tr} X|$ for X with nonzero trace, and $F(X) = \|X\|$ otherwise. Then F satisfies (i) – (iii). However if $\mathfrak{S} = \mathcal{I}^1$, then any mapping of the form $A \mapsto SAS^{-1}$ for an invertible (possibly non-unitary) S will satisfy $F(ABA) = F(\phi(A)\phi(B)\phi(A))$ for all $A, B \in \mathfrak{S}$.

Remark 4.3. If $\mathfrak{S}$ contains all matrices of rank-one, surjectivity assumption may be removed — all conclusions remain the same; the only difference is that in the first assertion, σ is a (possibly nonsurjective) field homomorphism.

Remark 4.4. Evidently, Theorem 4.1 can be used to treat many scalar functions on $M_n(\mathbb{F})$ including all the unitary similarity invariant norms ν, i.e., those norms ν satisfying $\nu(U^*AU) = \nu(A)$ for all $U, A \in M_n(\mathbb{F})$ with $U^*U = I$. One can also use the above result to treat non-scalar value functions. For example, denote by $W(T)$ the numerical range of a complex matrix defined by $W(T) = \{\mathbf{x}^*T\mathbf{x} : \mathbf{x} \in \mathbb{C}^n\}$. Suppose

$$W(ABA) = W(\phi(A)\phi(B)\phi(A)) \qquad \text{for all } A, B \in \mathfrak{S}.$$

Then $r(ABA) = r(\phi(A)\phi(B)\phi(A))$ for all $A, B \in \mathfrak{S}$. By Theorem 4.1, there is a unitary matrix U and a scalar function $\alpha : M_n(\mathbb{F}) \to \{\mu \in \mathbb{C} : |\mu| = 1\}$ such that ϕ has the form

$$A \mapsto \alpha(A) \cdot UAU^* \quad \text{or} \quad A \mapsto \alpha(A) \cdot UA^{\mathrm{t}}U^*.$$

Note that if X has rank-one, then $W(X)$ is an elliptical disk with foci 0 and $\operatorname{Tr} X$. We see that $\alpha(X)^3 = 1$ for all rank-one idempotents. One can then show that $\alpha(X) = \xi$ with $\xi^3 = 1$ for all $X \in \mathfrak{S}$ (see also [8]).

We can apply similar arguments to get other results. Moreover, we can use Theorem 3.3 and its corollary to get similar results on (complex) Hermitian matrices.

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