

PROPERTIES AND PRESERVERS OF THE PSEUDOSPECTRUM

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Dedicated to Professor Hans Schneider.

Abstract

The interplay between the algebraic and analytic properties of a matrix and the geometric properties of its pseudospectrum is investigated. It is shown that one can characterize Hermitian matrices, positive semi-definite matrices, orthogonal projections, unitary matrices, etc. in terms of the pseudospectrum. Also, characterizations are given to maps on matrices leaving invariant the pseudospectrum of the sum, difference, or product of matrix pairs. It is shown that such a map is always a unitary similarity transform followed by some simple operations such as adding a constant matrix, taking the matrix transpose, or multiplying by a scalar in $\{1, -1\}$.

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1 Introduction

Denote by M_n the set of $n \times n$ complex matrices equipped with the *operator norm* $\|\cdot\|$ induced by the usual vector norm $\|x\| = (x^*x)^{1/2}$ on \mathbf{C}^n , i.e.,

$$\|A\| = \max\{\|Ax\| : x \in \mathbf{C}^n, 0 < \|x\| \leq 1\}.$$

Let $\varepsilon > 0$. The *pseudospectrum* of a matrix $A \in M_n$ is defined by

$$\sigma_\varepsilon(A) = \{\mu \in \mathbf{C} : \text{there is } x \in \mathbf{C}^n, E \in M_n \text{ with } \|E\| \leq \varepsilon \text{ such that } (A + E)x = \mu x\}.$$

(Some authors use “ $\|E\| < \varepsilon$ ” instead of “ $\|E\| \leq \varepsilon$ ” in the definition of $\sigma_\varepsilon(A)$. One can easily adapt our results and proofs using this different definition.) The pseudospectrum of a matrix A for a given ε consists of all eigenvalues of matrices which are ε -close to A . Numerical algorithms which calculate the eigenvalues of a matrix give only approximate results due to rounding and other errors. These errors can be described with the matrix E . There are many interesting results concerning the pseudospectrum and its applications to applied problems; see [7]. Moreover, many

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researchers have derived efficient algorithms to generate pseudospectra of matrices; see [3] and its references.

In this paper, we show that the pseudospectrum can be used to study the algebraic and geometric properties of matrices; see Section 2. For example, we show that one can characterize Hermitian matrices, positive semidefinite matrices, orthogonal projections, unitary matrices, etc. in terms of pseudospectrum. Moreover, we study maps $\Phi : M_n \rightarrow M_n$ such that $\sigma_\varepsilon(A \circ B) = \sigma_\varepsilon(\Phi(A) \circ \Phi(B))$ for all $A, B \in M_n$, where $A \circ B = A + B, A - B$ or AB . We show that such a map is always a unitary similarity transform followed by some simple operations such as adding a constant matrix, taking the matrix transpose, or multiplying by a scalar in $\{1, -1\}$; see Section 3.

We will use the following equivalent definitions of pseudospectrum in our discussion; see [7].

- (1) $\sigma_\varepsilon(A) = \{\mu \in \mathbf{C} : \|(\mu I - A)x\| \leq \varepsilon \text{ for some unit vector } x \in \mathbf{C}^n\}$.
- (2) Denote by $s_1(A) \geq \dots \geq s_n(A)$ the singular values of $A \in M_n$. Then

$$\sigma_\varepsilon(A) = \{\mu \in \mathbf{C} : s_n(\mu I - A) \leq \varepsilon\}.$$

- (3) Using the convention that $\|(A - \lambda I)^{-1}\| = \infty$ for $\lambda \in \sigma(A)$, we have

$$\sigma_\varepsilon(A) = \{\mu \in \mathbf{C} : \|(A - \mu I)^{-1}\| \geq \varepsilon^{-1}\}.$$

The following properties are useful; see [7, Theorem 2.2 and 2.4].

Proposition 1.1 *Let $\varepsilon > 0$ and $A \in M_n$.*

- (a) *If $A = A_1 \oplus A_2$, then $\sigma_\varepsilon(A) = \sigma_\varepsilon(A_1) \cup \sigma_\varepsilon(A_2)$.*
- (b) *We have $\sigma(A) + D(0, \varepsilon) \subseteq \sigma_\varepsilon(A)$. The set equality holds if A is normal.*
- (c) *For any $c \in \mathbf{C}$, $\sigma_\varepsilon(A + cI) = c + \sigma_\varepsilon(A)$.*
- (d) *For any nonzero $c \in \mathbf{C}$, $\sigma_{|c|\varepsilon}(cA) = c\sigma_\varepsilon(A)$.*
- (e) *$\sigma_\varepsilon(A)$ is a nonempty compact subset of \mathbf{C} , and any bounded connected component of $\sigma_\varepsilon(A)$ has a nonempty intersection with $\sigma(A)$. Consequently, $\sigma_\varepsilon(A)$ has of at most n connected components.*

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In our discussion, we always assume that $n \geq 2$ to avoid trivial consideration. The following notation and definitions will be used.

M_n : the set of $n \times n$ complex matrices.

$\{E_{11}, E_{12}, \dots, E_{nn}\}$: the standard basis for M_n .

$\{e_1, \dots, e_n\}$: the standard basis for \mathbf{C}^n .

$D(a, r) = \{\mu \in \mathbf{C} : |\mu - a| \leq r\}$, where $a \in \mathbf{C}$ and $r \geq 0$.

2 The pseudospectrum and matrix properties

Proposition 2.1 *Suppose $\varepsilon > 0$ and $A = \begin{pmatrix} a & b \\ 0 & c \end{pmatrix} \in M_2$. Then*

$$\sigma_\varepsilon(A) = \left\{ \mu \in \mathbf{C} : \sqrt{(|\mu - a| + |\mu - c|)^2 + |b|^2} - \sqrt{(|\mu - a| - |\mu - c|)^2 + |b|^2} \leq 2\varepsilon \right\}.$$

Consequently, $b = 0$ if and only if $\sigma_\varepsilon(A) = D(a, \varepsilon) \cup D(c, \varepsilon)$. If $a = c = 0$, then

$$\sigma_\varepsilon(A) = \left\{ \mu : |\mu| \leq \sqrt{\varepsilon(\varepsilon + |b|)} \right\}.$$

Proof. Let s_1 and s_2 be the singular values of $\mu I_2 - A$. Then $s_1 s_2 = |\det(A)| = |(\mu - a)(\mu - c)|$ and

$$s_1^2 + s_2^2 = \operatorname{tr}((\mu I_2 - A)^*(\mu I_2 - A)) = |\mu - a|^2 + |b|^2 + |\mu - c|^2.$$

Thus, $(s_1 \pm s_2)^2 = (|\mu - a| \pm |\mu - c|)^2 + |b|^2$ so that

$$2s_2 = \sqrt{(|\mu - a| + |\mu - c|)^2 + |b|^2} - \sqrt{(|\mu - a| - |\mu - c|)^2 + |b|^2}.$$

The description of $\sigma_\varepsilon(A)$ follows.

The last two assertions can be verified readily. \square

By Proposition 1.1(b), if A is normal, then $\sigma_\varepsilon(A) = \sigma(A) + D(0, \varepsilon)$. By Proposition 2.1, for $\varepsilon > 0$ and $A \in M_2$, if $\sigma_\varepsilon(A)$ is the union of two disks, which may be identical, with radius ε , then A is normal. We do not need to know $\sigma(A)$ in advance to conclude that $A \in M_2$ is normal in terms of $\sigma_\varepsilon(A)$. However, the situation for higher dimensions is more delicate. In fact, contrary to the belief of some authors (see [7, Theorem 2.4]), the converse of Proposition 1.1(b) is not true as shown in the following example.

Example 2.2 Let $\varepsilon = 2$, $w = e^{i2\pi/3}$, and $A = A_1 \oplus A_2$ with $A_1 = \operatorname{diag}(1, w, w^2)$ and $A_2 = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$, where $b > 0$ satisfies $\sigma_\varepsilon(A_2) = \{\mu \in \mathbf{C} : |\mu| \leq \sqrt{2(2+b)}\} \subseteq D(0, (\sqrt{13} + 1)/2)$. Then A is not normal and

$$\sigma_\varepsilon(A) = \sigma_\varepsilon(A_1) \cup \sigma_\varepsilon(A_2) = \sigma_\varepsilon(A_1) = \sigma(A) + D(0, \varepsilon).$$

It is not hard to see that the problem in Example 2.2 occurs because $\sigma_\varepsilon(A_2)$ is a subset of $\sigma(A) + D(0, \varepsilon)$. As a result, $\sigma_\varepsilon(A)$ fails to detect that 0 is not a *reducing eigenvalue* of A . Recall that μ is an reducing eigenvalue of $B \in M_n$ if B is unitarily similar to $\mu I_k \oplus \hat{B}$, where k is the algebraic multiplicity of the eigenvalue μ of B ; a matrix $B \in M_n$ is normal if and only if each eigenvalue of B is reducing. Theorem 2.5 below shows that a stronger condition on $\sigma_\varepsilon(A)$ is needed to conclude that A is normal. We first prove the following lemmas.

Lemma 2.3 Let $\varepsilon > 0$. Suppose $A = \begin{pmatrix} A_1 & * \\ 0 & A_2 \end{pmatrix} \in M_n$ with $A_1 \in M_k$. Then $\sigma_\varepsilon(A_1) \subseteq \sigma_\varepsilon(A)$.

Proof. Suppose $\mu \in \sigma_\varepsilon(A_1)$. Then there is a unit vector $x \in \mathbf{C}^k$ such that $\|(\mu I - A_1)x\| \leq \varepsilon$. Let $\tilde{x} = \begin{pmatrix} x \\ 0 \end{pmatrix} \in \mathbf{C}^n$. Then $\|(\mu I - A)\tilde{x}\| \leq \varepsilon$ so that $\mu \in \sigma_\varepsilon(A)$. \square

Lemma 2.4 *Let $\varepsilon > 0$, $A \in M_n$, and $\mu \in \sigma(A)$. If $D(\mu, \varepsilon)$ contains a boundary point of $\sigma_\varepsilon(A)$, then μ is a reducing eigenvalue of A .*

Proof. Suppose μ is not an reducing eigenvalue of A . We may assume that A is in upper triangular form with the leading 2×2 submatrix equal to $A_1 = \begin{pmatrix} \mu & d \\ 0 & \mu_2 \end{pmatrix}$ with $d > 0$; see the lemma in [6]. By Proposition 2.1, $D(\mu, \varepsilon)$ lies in the interior of $\sigma_\varepsilon(A_1)$, which is a subset of $\sigma_\varepsilon(A)$ by Lemma 2.3. \square

Theorem 2.5 *Let $\varepsilon > 0$, $A \in M_n$, and $\mu_1, \dots, \mu_m \in \mathbf{C}$ be such that $\sigma_\varepsilon(A) = \cup_{j=1}^m D(\mu_j, \varepsilon)$ and each $D(\mu_j, \varepsilon)$ contains a boundary point of $\sigma_\varepsilon(A)$. Moreover, suppose that $D(\mu, \varepsilon) \not\subseteq \sigma_\varepsilon(A)$ for any $\mu \notin \{\mu_1, \dots, \mu_m\}$, then A is unitarily similar to $\mu_1 I \oplus \dots \oplus \mu_m I$.*

Proof. Since $\sigma(A) + D(0, \varepsilon) \subseteq \sigma_\varepsilon(A)$, we see that $\sigma(A) \subseteq \{\mu_1, \dots, \mu_m\}$ under the hypothesis of the theorem. If $\mu_j \in \sigma(A)$, then μ_j is a reducing eigenvalue by Lemma 2.4. Clearly, each μ_j is an eigenvalue of A . Otherwise, $\sigma_\varepsilon(A) \subseteq \cup_{\ell \neq j} D(\mu_\ell, \varepsilon)$ cannot contain boundary point of $D(\mu_j, \varepsilon)$. \square

Corollary 2.6 *Let $\varepsilon > 0$, $A \in M_n$ and $\mu \in \mathbf{C}$.*

- (a) *We have $A = \mu I$ if and only if $\sigma_\varepsilon(A) = D(\mu, \varepsilon)$.*
- (b) *We have $A = \mu P$ for a nontrivial orthogonal projection P if and only if $\sigma_\varepsilon(A) = D(\mu, \varepsilon) \cup D(0, \varepsilon)$.*
- (c) *The matrix A is positive semidefinite (respectively, positive definite) if and only if each element $\mu \in \sigma_\varepsilon(A)$ satisfies $|\operatorname{Im}(\mu)| \leq \varepsilon$ and $\operatorname{Re}(\mu) \geq -\varepsilon$ (respectively, $\operatorname{Re}(\mu) > -\varepsilon$).*
- (d) *Suppose $t \in [0, 2\pi)$ and $\xi \in \mathbf{C}$. Then $e^{it}A + \xi I$ is Hermitian if and only if*

$$\sigma_\varepsilon(A) \subseteq \{\mu \in \mathbf{C} : |\operatorname{Im}(e^{-it}\mu - \xi)| \leq \varepsilon\}.$$

- (e) *Suppose $\varepsilon \in (0, 1/2)$. Then A is unitary if and only if $\sigma_\varepsilon(A)$ is the union of circular disks each has radius ε with centers lying on the unit circle.*

Note that the assumption $\varepsilon \in (0, 1/2)$ in (e) is important. Otherwise, we may have the same problem as in Example 2.2, namely, we have $A = A_1 \oplus A_2$, where $A_1 = \operatorname{diag}(1, w, \dots, w^{n-1})$ with $w = e^{i2\pi/n}$ for a sufficiently large n , and $A_2 = \begin{pmatrix} 0 & b \\ 0 & 0 \end{pmatrix}$ for a sufficiently small $b > 0$ so that

$$\sigma_\varepsilon(A_2) = D\left(0, \sqrt{\varepsilon(\varepsilon + b)}\right) \subseteq \cup_{j=1}^n D(w^j, \varepsilon) = \sigma_\varepsilon(A_1) = \sigma_\varepsilon(A).$$

Proof. We give details of the proof of (d). The implication (\Leftarrow) is clear. Consider the converse. For simplicity, assume that $e^{it} = 1$ and $\xi = 0$. Since $D(\lambda, \varepsilon) \subseteq \sigma_\varepsilon(A)$ for any $\lambda \in \sigma(A)$, we see that $\sigma(A) \subseteq \mathbf{R}$. Suppose A is not normal. Then there is a unitary matrix such that UAU^* is in upper triangular form so that the leading 2×2 submatrix $B = \begin{pmatrix} a_{11} & a_{12} \\ 0 & a_{22} \end{pmatrix}$ has a nonzero a_{12} entry; e.g.,

see [6]. Recall that $a_{11}, a_{22} \in \mathbf{R}$. Consequently, for $z = a_{11} + i\varepsilon$, $s_n(zI - A) < \varepsilon$ since $|a_{12}| \neq 0$. Thus, there is sufficiently small $d > 0$ such that for $\tilde{z} = a_{11} + i(\varepsilon + d)$, we have $\tilde{z} \in \sigma_\varepsilon(A)$ and $\text{Im}(\tilde{z}) > \varepsilon$, which is a contradiction. Thus, A is normal with real eigenvalues, i.e., $A = A^*$.

(c) follows from (d). \square

For $A \in M_n$, denote by $r_\varepsilon(A) = \max\{|\lambda| : \lambda \in \sigma_\varepsilon(A)\}$. The following proposition on $r_\varepsilon(A)$ for $A \in M_2$ is useful for the proofs in the next section.

Proposition 2.7 *Let $A = aE_{11} + bE_{12} \in M_n$ for some nonzero $a, b \in \mathbf{C}$. Then*

$$r_\varepsilon(A) = \frac{1}{2} \left\{ \left[|a|^2 + 4\varepsilon^2 + 4\varepsilon\sqrt{|a|^2 + |b|^2} \right]^{1/2} + |a| \right\}.$$

The equality is attained at a unique value $z \in \sigma_\varepsilon(A)$ of the form $z = ta$ for a positive t .

Proof. We may replace A by $e^{is}A$ for some $s \in \mathbf{R}$ and assume that $a > 0$. We know that $D(a, \varepsilon) \subseteq \sigma_\varepsilon(A)$. Let $z \in \sigma_\varepsilon(A) \cap D(a, \infty)$. Then $zI - A$ has singular values $s_1 \geq \dots \geq s_n$ such that $s_2 = \dots = s_{n-1} = z$ and s_1, s_n are the singular values of the matrix $B_z = \begin{pmatrix} z - a & -b \\ 0 & z \end{pmatrix}$. Thus, $s_1^2 + s_n^2 = \text{tr}(zI - B)^*(zI - B) = |z - a|^2 + |b|^2 + |z|^2$ and $s_1 s_n = |\det(zI - B)| = |(z - a)z|$. It follows that

$$(s_1 \pm s_n)^2 = (|z| \pm |z - a|)^2 + |b|^2$$

and

$$s_n = \frac{1}{2} \left\{ \sqrt{(|z| + |z - a|)^2 + |b|^2} - \sqrt{(|z| - |z - a|)^2 + |b|^2} \right\}. \quad (1)$$

It is known that $D(a, \varepsilon) \subseteq \sigma_\varepsilon(A)$. It follows that $z = ta \in \sigma_\varepsilon(A)$ for some $t > 1$. Note that for $z = ta$ with $t > 1$, (1) simplifies to

$$s_n = \frac{1}{2} \left\{ \sqrt{(2z - a)^2 + |b|^2} - \sqrt{a^2 + |b|^2} \right\}. \quad (2)$$

Thus, $s_n \leq \varepsilon$ if and only if

$$z \leq \frac{1}{2} \left\{ [a^2 + 4\varepsilon^2 + 4\varepsilon\sqrt{a^2 + |b|^2}]^{1/2} + a \right\}, \quad (3)$$

where the equality holds for a suitable choice of $\hat{z} = ta$ for some $t > 1$.

Next, we show that if $z \in \mathbf{C}$ satisfies $|z| \geq |\hat{z}|$ and $z \neq \hat{z}$, then $s_n(zI - A) > \varepsilon$. Our result will follow. To prove the above claim, note that if z is positive and $z > \hat{z}$, then (3) will be violated, and thus $z \notin \sigma_\varepsilon(A)$. Thus, if $z > \hat{z}$, then $s_n(zI - A) > \varepsilon$. Next, we show that if $z \geq \hat{z}$, then for $e^{is} \neq 1$, $s_n(ze^{is}I - A) > s_n(zI - A) \geq \varepsilon$. Our claim will be established. To this end, note that $|ze^{is} - a| > z - a > 0$ so that $|ze^{is} - a| = u(z - a)$ for some $u > 1$; the matrices $B_{ze^{is}}$ and $\tilde{B} = \begin{pmatrix} |ze^{is} - a| & |b| \\ 0 & z \end{pmatrix}$ have the same singular values. Note that none of B_z nor \tilde{B} can be a multiple

of a unitary; hence each of them has distinct singular values. We may suppose B_z has singular values $\mu_1 > \mu_2 \geq 0$, and \tilde{B} has singular values $u_1\mu_1 > u_2\mu_2 \geq 0$ for some $u_1, u_2 \geq 0$. Then

$$|u \det(B_z)| = |\det(\tilde{B})| = u_1 u_2 \mu_1 \mu_2 = u_1 u_2 |\det(B_z)|$$

so that $u = u_1 u_2$;

$$u_1^2 \mu_1^2 + u_2^2 \mu_2^2 = |ze^{is} - a|^2 + |b|^2 + z^2 = u^2(z - a)^2 + |b|^2 + z^2$$

and

$$\mu_1^2 + \mu_2^2 = (z - a)^2 + |b|^2 + z^2$$

so that

$$(u_1^2 - 1)\mu_1^2 + (u_2^2 - 1)\mu_2^2 = (u^2 - 1)(z - a)^2 = (u_1^2 u_2^2 - 1)(z - a)^2.$$

If $u_2 \leq 1$, then $u_1 = u/u_2 > 1$. By the fact that $\mu_1 > \mu_2$ and $\mu_1 > z - a$, we have

$$(u_1^2 - 1)\mu_1^2 = (u_1^2 u_2^2 - 1)(z - a)^2 + (1 - u_2^2)\mu_2^2 < (u_1^2 u_2^2 - 1 + 1 - u_2^2)\mu_1^2 = (u_1^2 - 1)u_2^2 \mu_1^2,$$

which is a contradiction. Thus, we have $s_2(\tilde{B}) = u_2 \mu_2 > \mu_2 = s_2(B_z)$. Our claim follows. \square

3 Preservers

3.1 Sums and differences of matrices

In this subsection, we prove the following.

Theorem 3.1 *Let $\varepsilon > 0$ and $\Phi : M_n \rightarrow M_n$. Then $\sigma_\varepsilon(\Phi(A) - \Phi(B)) = \sigma_\varepsilon(A - B)$ for all $A, B \in M_n$ if and only if there are $U, S \in M_n$ such that U is unitary and Φ has the form*

$$A \mapsto UAU^* + S \quad \text{or} \quad A \mapsto UA^t U^* + S.$$

From this result, we can deduce the following.

Theorem 3.2 *Let $\varepsilon > 0$, and $\Phi : M_n \rightarrow M_n$. The following are equivalent.*

- (a) Φ is linear and satisfies $\sigma_\varepsilon(\Phi(A)) = \sigma_\varepsilon(A)$ for all $A \in M_n$.
- (b) Φ is additive and satisfies $\sigma_\varepsilon(\Phi(A)) = \sigma_\varepsilon(A)$ for all $A \in M_n$.
- (c) Φ satisfies $\sigma_\varepsilon(\Phi(A) + \Phi(B)) = \sigma_\varepsilon(A + B)$ for all $A, B \in M_n$.
- (d) There is a unitary matrix $U \in M_n$ such that Φ has the form

$$A \mapsto UAU^* \quad \text{or} \quad A \mapsto UA^t U^*.$$

Proof. The implications (d) \Rightarrow (a) \Rightarrow (b) \Rightarrow (c) are clear. To prove (c) \Rightarrow (d), note that $\sigma_\varepsilon(\Phi(0) + \Phi(0)) = \sigma_\varepsilon(0 + 0) = D(0, \varepsilon)$. Thus, $\Phi(0) = 0$. Moreover, for any $B \in M_n$,

$$\sigma_\varepsilon(\Phi(B) + \Phi(-B)) = \sigma_\varepsilon(B - B) = D(0, \varepsilon).$$

Thus, $\Phi(B) + \Phi(-B) = 0$, i.e., $\Phi(-B) = -\Phi(B)$. Consequently, $\sigma_\varepsilon(\Phi(A) - \Phi(B)) = \sigma_\varepsilon(\Phi(A) + \Phi(-B)) = \sigma_\varepsilon(A - B)$ for any $A, B \in M_n$. Applying the result of Theorem 3.1 and the fact that $\Phi(0) = 0$, we see that Φ has the asserted form in Theorem 3.1 with $S = 0$. The result follows. \square

We need a few more definitions and notation to prove Theorem 3.1. For $A \in M_n$, let

$r(A)$ be the spectral radius of A ,

$W(A) = \{x^*Ax : x \in \mathbf{C}^n, x^*x = 1\}$ be the numerical range of A , and

$w(A) = \max\{|\mu| : \mu \in W(A)\}$ be the numerical radius of A .

If $A \in H_n$, the set of Hermitian matrices in M_n , has eigenvalues $a_1 \geq \dots \geq a_n$, then

$$W(A) = [a_n, a_1] \quad \text{and} \quad r(A) = w(A) = \|A\| = \max\{|a_1|, |a_n|\}.$$

Proof of Theorem 3.1 The implication (\Leftarrow) is clear. To prove the converse, we may set $\Phi(0) = S$ and replace Φ by the map $A \mapsto \Phi(A) - S$, and assume that $\Phi(0) = 0$, $\sigma_\varepsilon(\Phi(A)) = \sigma_\varepsilon(A)$ for all $A \in M_n$, and $\sigma_\varepsilon(\Phi(A) - \Phi(B)) = \sigma_\varepsilon(A - B)$ for all $A, B \in M_n$. By Corollary 2.6 (d), $\Phi(H_n) \subseteq H_n$. Note that for any $A \in H_n$, $\sigma_\varepsilon(A) = \cup_{\lambda \in \sigma(A)} D(\lambda, \varepsilon)$, $r_\varepsilon(A) = r(A) + \varepsilon$ and $r(A) = w(A) = \|A\|$. Since $\sigma_\varepsilon(\Phi(A) - \Phi(B)) = \sigma_\varepsilon(A - B)$, we see that $\|\Phi(A) - \Phi(B)\| = \|A - B\|$. By the result in [2], Φ is a real linear map. Moreover, $W(\Phi(A)) = W(A)$. Thus, there is unitary $U \in M_n$ such that the restriction of Φ on Hermitian matrices has the form

$$A \mapsto UAU^* \quad \text{or} \quad A \mapsto UA^tU^*,$$

see [4, 5] and their references.

Similarly, we can show that $\Phi(iH_n) \subseteq iH_n$, and there is a unitary $V \in M_n$ such that the restriction of Φ on skew-Hermitian matrices has the form

$$A \mapsto VAV^* \quad \text{or} \quad A \mapsto VA^tV^*.$$

Now, if $A \in M_n$, then $A = H + iG$ for some $H, G \in H_n$. Suppose $\Phi(A) = R + iS$ for some $R, S \in H_n$. Then

$$\sigma_\varepsilon(R + iS - \Phi(H)) = \sigma_\varepsilon(\Phi(A) - \Phi(H)) = \sigma_\varepsilon(A - H) = \sigma_\varepsilon(iG) \subseteq i\{\mu \in \mathbf{C} : |\text{Im}(\mu)| \leq \varepsilon\}$$

implies that $R + iS - \Phi(H)$ is skew-Hermitian, i.e., $R = \Phi(H)$. Similarly, we can show that $iS = \Phi(G)$.

We **claim** that $\Phi(iH) = i\Phi(H)$ for any Hermitian matrix H . Once this is proved, Φ will have the asserted form.

To prove our **claim**, consider $A = xx^* + ixx^*$ for any unit vector $x \in \mathbf{C}^n$. Then

$$\Phi(A) = \Phi(xx^*) + \Phi(ixx^*),$$

where

$$(1) \quad \Phi(xx^*) = Uxx^*U^* \text{ or } (Uxx^*U^*)^t, \quad \text{and} \quad (2) \quad \Phi(ixx^*) = i(Vxx^*V^*) \text{ or } i(Vxx^*V^*)^t.$$

Since $\sigma_\varepsilon(\Phi(A)) = \sigma_\varepsilon(A) = D(1+i, \varepsilon) \cup D(0, \varepsilon)$, which, together with Corollary 2.6 (b), we have $\Phi(A) = (1+i)P$ for some nontrivial projection P , it follows that $\Phi(ixx^*) = i\Phi(xx^*)$. As this is true for any unit vector x and the restriction of Φ on Hermitian matrices (respectively, skew-Hermitian matrices) is real linear, the result follows. \square

3.2 Products of matrices

In this subsection, we prove the following.

Theorem 3.3 *Let $\varepsilon > 0$, and $\Phi : M_n \rightarrow M_n$. Then*

$$\sigma_\varepsilon(\Phi(A)\Phi(B)) = \sigma_\varepsilon(AB) \quad \text{for all } A, B \in M_n$$

if and only if there is a unitary $U \in M_n$ and $\xi \in \{-1, 1\}$ such that Φ has the form

$$A \mapsto \xi U^* A U.$$

By Theorem 3.3, one easily gets the following.

Corollary 3.4 *Let $\varepsilon > 0$. A multiplicative map $\Phi : M_n \rightarrow M_n$ satisfies*

$$\sigma_\varepsilon(\Phi(A)) = \sigma_\varepsilon(A) \quad \text{for all } A \in M_n$$

if and only if there is a unitary $U \in M_n$ such that Φ has the form

$$A \mapsto U^* A U.$$

To prove Theorem 3.3, we need the following lemma; see [1, Theorem 2.1].

Lemma 3.5 *Suppose $n \geq 3$. Suppose $\Phi : M_n \mapsto M_n$ satisfies $\Phi(A)\Phi(B) = 0$ if and only if $AB = 0$. Then there exists a field monomorphism $\tau : \mathbf{C} \rightarrow \mathbf{C}$, a mapping $\mu : M_n \mapsto \mathbf{C} \setminus \{0\}$, and $S \in M_n$ such that Φ has the form:*

$$A \mapsto \mu(A)S(\tau(a_{ij}))S^{-1} \quad \text{for all rank one matrix } A = (a_{ij}) \in M_n.$$

Proof of Theorem 3.3 The implication “ \Leftarrow ” is clear. We consider the converse. By our assumption on Φ , $\sigma_\varepsilon(AB) = D(0, \varepsilon)$ if and only if $\sigma_\varepsilon(\Phi(A)\Phi(B)) = D(0, \varepsilon)$. Thus, $AB = 0$ if and only if $\Phi(A)\Phi(B) = 0$.

Case 1 Suppose $n \geq 3$. Then Φ has the form described in Lemma 3.5. Suppose u_1, \dots, u_n are the columns of S and v_1^*, \dots, v_n^* are the rows of S^{-1} . If $A = E_{jj}$, then

$$D(0, \varepsilon) \cup D(1, \varepsilon) = \sigma_\varepsilon(E_{jj}) = \sigma_\varepsilon(A^2) = \sigma_\varepsilon(\Phi(A)^2) = \sigma_\varepsilon(\mu(A)^2 u_j v_j^*).$$

Thus, $\mu(A)^2 u_j v_j^* = x_j x_j^*$ for some unit vector $x_j \in \mathbf{C}^n$ by Corollary 2.6 (b). Consequently, $u_j = d_j v_j$ for some $d_j \in \mathbf{C}$. Hence, $S^* = DS^{-1}$ so that $S^*S = D$, where $D = \text{diag}(d_1, \dots, d_n)$. As a result, D has positive diagonal entries. Now, for $A = (E_{11} + E_{1j} + E_{j1} + E_{jj})/2$, we have $A^2 = A$ and

$$D(0, \varepsilon) \cup D(1, \varepsilon) = \sigma_\varepsilon(A^2) = \sigma_\varepsilon(\Phi(A)^2) = \sigma_\varepsilon(2^{-1}(\mu(A)^2 v_1 d_1 v_1^* + v_1 d_1 v_1^* + v_j d_j v_j^* + v_j d_j v_j^*)).$$

By Corollary 2.6 (b), $\Phi(A)^2$ is a rank one orthogonal projection. It follows that $\mu(A)^2 > 0$ and $d_1 = d_j$. Thus, we see that $D = d_1 I_n$. and $S^* = d_1 S^{-1}$, i.e., S is a multiple of a unitary matrix. Replacing S by γS for a suitable $\gamma > 0$, we may assume that S is unitary.

Next, we show that $|\tau(a)| = |a|$ for any $a \in \mathbf{C}$. To this end, let $A = E_{11} + aE_{1n}$. Then $\Phi(A) = \mu(A)S(E_{11} + \tau(a)E_{1n})S^{-1}$. For any $1 < k \leq n$,

$$|\mu(A)\mu(E_{1k})|\sigma_\varepsilon(SE_{1k}S^{-1}) = \sigma_\varepsilon(\Phi(A)\Phi(E_{1k})) = \sigma_\varepsilon(AE_{1k}) = \sigma_\varepsilon(E_{1k})$$

and

$$|\tau(a)\mu(A)\mu(E_{nk})|\sigma_\varepsilon(SE_{1k}S^{-1}) = \sigma_\varepsilon(\Phi(A)\Phi(E_{nk})) = \sigma_\varepsilon(AE_{nk}) = |a|\sigma_\varepsilon(E_{1k}).$$

It follows that $|\tau(a)| = |\mu(E_{1k})/\mu(E_{nk})||a|$. Since $\tau(1) = 1$, we see that $|\mu(E_{1k})| = |\mu(E_{nk})|$, and hence $|\tau(a)| = |a|$. It is well known that τ must either be the identity or the complex conjugation; see [8] for example. Suppose $\tau(a) = \bar{a}$ for all $a \in \mathbf{C}$. Let $A = \varepsilon(10e^{i\pi/8}E_{11} + e^{i3\pi/8}\sum_{j=2}^n E_{jj})$. Then

$$\sigma_\varepsilon(A^2) = D(100e^{i\pi/4}\varepsilon, \varepsilon) \cup D(e^{i6\pi/8}, \varepsilon)$$

and

$$\sigma_\varepsilon(\Phi(A)^2) = D(\mu(A)^2 100e^{-i\pi/4}\varepsilon, \varepsilon) \cup D(\mu(A)^2 e^{-i6\pi/8}, \varepsilon)$$

so that $\sigma_\varepsilon(\Phi(A)^2) \neq \sigma_\varepsilon(A^2)$ for any choice of $\mu(A) \in \mathbf{C}^*$. Thus, we see that τ is the identity map.

We may now replace Φ by the map $A \mapsto S^{-1}\Phi(A)S$ and assume that $S = I_n$. Now, for any unit vector $x \in \mathbf{C}^n$ and $A = xx^*$, we have

$$D(0, \varepsilon) \cup D(1, \varepsilon) = \sigma_\varepsilon(A^2) = \sigma_\varepsilon(\Phi(A)^2) = \sigma_\varepsilon(\mu(A)^2 A^2) = \sigma_\varepsilon(\mu(A)^2 A).$$

Thus, $\mu(xx^*)^2 = 1$ and $\mu(xx^*) \in \{1, -1\}$. Assume that $\mu(E_{11}) = 1$. Otherwise, replace Φ by the map $A \mapsto -\Phi(A)$. Then for any unit vector $x \in \mathbf{C}^n$ with $e_1^t x \neq 0$, we have

$$\sigma_\varepsilon(xx^*E_{11}) = \sigma_\varepsilon((\Phi(xx^*)\Phi(E_{11}))) = \sigma_\varepsilon(\mu(xx^*)xx^*E_{11}).$$

Thus, $\mu(xx^*) = 1$. Now, for a unit vector $y \in \mathbf{C}^n$ with $e_1^t y = 0$, we can find a unit vector so that $e_1^t x \neq 0$ and $x^t y \neq 0$ so that $\mu(xx^*) = 1$, and

$$\sigma_\varepsilon(xy^*xx^*) = \sigma_\varepsilon(\Phi(xy^*)\Phi(xx^*)) = \sigma_\varepsilon(\mu(xy^*)xy^*xx^*) = \sigma_\varepsilon(y^*x\mu(xy^*)xx^*).$$

By Corollary 2.6, we see $\mu(xy^*) = 1$. Next, we consider

$$\sigma_\varepsilon(yy^*xy^*) = \sigma_\varepsilon(\Phi(yy^*)\Phi(xy^*)) = \sigma_\varepsilon(\mu(yy^*)yy^*xy^*) = \sigma_\varepsilon(y^*x\mu(yy^*)yy^*),$$

which shows $\mu(yy^*) = 1$. Using $\sigma_\varepsilon(yx^*yy^*)$ in a similar way we have $\mu(yx^*) = 1$.

So for any unit vectors $x, y \in \mathbf{C}^n$ and x is not orthogonal to y , we have $\mu(xy^*) = 1$. If $x^*y = 0$, find a unit vector u such that $x^*u \neq 0$. Then,

$$\sigma_\varepsilon(xu^*yx^*) = \sigma_\varepsilon(\mu(yu^*)(yx^*)(u^*y)xx^*).$$

Since $\mu(yu^*) = 1$ (from above), we have $\mu(yx^*) = 1$. Applying a similar argument to $\sigma_\varepsilon(yu^*xy^*)$, we see that $\mu(xy^*) = 1$.

So, we have shown $\Phi(uv^*) = uv^*$ for any unit vectors $u, v \in \mathbf{C}^n$. Now, for any $A \in M_n$, let $B = \Phi(A)$. Suppose $A = UDV^*$ so that U, V are unitary and $D = \text{diag}(a_1, \dots, a_n)$ with $a_1 \geq \dots \geq a_n$. Then for each j ,

$$\sigma_\varepsilon(BVE_{jj}U^*) = \sigma_\varepsilon(AVE_{jj}U^*) = D(a_j, \varepsilon) \cup (0, \varepsilon).$$

We see that $BVE_{jj}U^*$ is Hermitian with eigenvalues $a_j, 0, \dots, 0$, and so is U^*BVE_{jj} . Hence, $U^*BV = D$, i.e., $B = A$.

Case 2 Suppose $n = 2$. We divide the proofs into several steps.

Step 1 If $x \in \mathbf{C}^2$ is a unit vector, then $\Phi(xx^*) = \pm yy^*$ for some unit vector $y \in \mathbf{C}^2$.

Invoking Corollary 2.6, $\sigma_\varepsilon(xx^*xx^*) = \sigma_\varepsilon(xx^*) = D(0, \varepsilon) \cup D(1, \varepsilon)$. So $\sigma_\varepsilon(\Phi(xx^*)^2)$ is the same. Using the reverse direction of Corollary 2.6, $\Phi(xx^*)^2 = yy^*$ for a unit vector $y \in \mathbf{C}$. It follows that $\Phi(xx^*) = \pm yy^*$.

Step 2 Assume $\Phi(E_{11}) = E_{11}$, then $(\Phi(I), \Phi(E_{22})) = (E_{11} + \xi E_{22}, \xi E_{22})$ with $\xi \in \{1, -1\}$. Moreover, there is a diagonal unitary D such that $D^*\Phi(X)D = X$ for $X = \{E_{12}, E_{21}\}$.

Since $\sigma_\varepsilon(E_{11}^2) = \sigma_\varepsilon(E_{11}I) = \sigma_\varepsilon(IE_{11})$, we have $\sigma_\varepsilon(\Phi(E_{11})) = \sigma_\varepsilon(\Phi(E_{11})\Phi(I)) = \sigma_\varepsilon(\Phi(I)\Phi(E_{11}))$, we see that $\Phi(I) = \text{diag}(1, \xi)$. Then consider $\sigma_\varepsilon(E_{22}^2) = \sigma_\varepsilon(E_{22}I) = \sigma_\varepsilon(IE_{22})$. We get the first assertion.

Next, consider $\sigma_\varepsilon(E_{ij}) = \sigma_\varepsilon(IE_{ij}) = \sigma_\varepsilon(E_{ij}I)$ for $\{i, j\} = \{1, 2\}$, and $\sigma_\varepsilon(E_{12}E_{21}) = \sigma_\varepsilon(E_{21}E_{12})$, we get the second assertion.

Step 3 Assume that $A = \frac{1}{2} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \mapsto \pm yy^*$ where y is a unit vector. Then $\Phi(X) = X$ for $X = E_{22}, I$.

Consider $\sigma_\varepsilon\left(\frac{1}{2} \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix}\right) = \sigma_\varepsilon(E_{11}A) = \sigma_\varepsilon(\Phi(E_{11})\Phi(A)) = \sigma_\varepsilon\left(\begin{pmatrix} |y_1|^2 & y_1\bar{y}_2 \\ y_2\bar{y}_1 & |y_2|^2 \end{pmatrix}\right)$. By Proposition 2.7, $r_\varepsilon = t_1 \cdot |y_1|^2$ for some positive t . Since $E_{22}A$ is unitarily similar to $E_{11}A$, we see that $\sigma_\varepsilon(E_{11}A) = \sigma_\varepsilon(\Phi(E_{22})\Phi(A))$. So $t_1|y_1|^2 = r_\varepsilon = t_2|y_2|^2$. Since t_1, t_2 are positive, $\Phi(E_{11})$ and $\Phi(E_{22})$ must have the same sign. Therefore, $\Phi(E_{22}) = E_{22}$. Similarly, $\Phi(I) = I$.

Step 4 Under the assumptions of Step 3, we have $\Phi(X) = X$ for all $X \in M_2$.

Assume that $A = (a_{ij})$. Using Proposition 2.7, $\Phi(A) = (t_{ij}a_{ij})$. Take a matrix $X = \begin{pmatrix} x_1 & x_2 \\ 0 & 0 \end{pmatrix}$ where the first row is orthogonal to the second column of A . Then $XA = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$ and $\sigma_\varepsilon(XA) =$

$\sigma_\varepsilon(\Phi(X)\Phi(A))$. So clearly $\Phi(X)\Phi(A) = \begin{pmatrix} * & 0 \\ 0 & 0 \end{pmatrix}$, meaning $(t_{12}a_{12}, t_{22}a_{22})$ is parallel to (a_{12}, a_{22}) . Therefore $t_{12} = t_{22}$. Choosing matrices X such that the entries are orthogonal to the other columns and rows of A , we see that all $t_{ij} = t$ are the same.

Consider $\sigma_\varepsilon(E_{11}A) = \sigma_\varepsilon(E_{11}\Phi(A))$. Since $E_{11}\Phi(A)$ is a multiple of $E_{11}A$ we see from Proposition 2.7 that $t = 1$. Therefore $\Phi(A) = A$. \square

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