# Linear maps transforming $H$-Unitary Matrices

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**Dedicated to Professor Yung-Chow Wong on the occasion of his 90-th birthday**

## Abstract

Let $H_1$ be an $n \times n$ invertible Hermitian matrix, and let $\mathbf{U}(H_1)$ be the group of $n \times n$ $H_1$-unitary matrices, i.e., matrices $A$ satisfying $A^* H_1 A = H_1$. Suppose $H_2$ is an $m \times m$ invertible Hermitian matrix. We show that a linear transformation $\phi : M_n \to M_m$ satisfies $\phi(\mathbf{U}(H_1)) \subseteq \mathbf{U}(H_2)$ if and only if there exist invertible matrices $S \in M_m$, $U, V \in \mathbf{U}(H_2)$ such that

$$S^* H_2 S = [(I_a \oplus -I_b) \otimes H_1] \oplus [(I_c \oplus -I_d) \otimes (H_1^{-1})^t],$$

and $\phi$ has the form

$$A \mapsto US[(I_{a+b} \otimes A) \oplus (I_{c+d} \otimes A^t)]S^{-1}V,$$

where $a$, $b$, $c$ and $d$ are nonnegative integers satisfying $(a+b+c+d)n = m$. Assume $H_1$ has inertia $(p,q)$ and $H_2$ has inertia $(r,s)$. Then there is a linear transformation mapping $\mathbf{U}(H_1)$ into $\mathbf{U}(H_2)$ if and only if there are nonnegative integers $u$ and $v$ such that $(r,s) = u(p,q) + v(q,p)$. These results generalize those of Marcus, Cheung and Li.

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# 1 Introduction

Let $M_n$ be the algebra of $n \times n$ matrices. Suppose $H$ is an invertible Hermitian matrix. A matrix $A \in M_n$ is *$H$-unitary* if $A^*HA = H$. Let $\mathbf{U}(H)$ be the set of $H$-unitary matrices. One readily checks that $\mathbf{U}(H)$ is a group, and $\mathbf{U}(I_n)$ is the usual unitary group. The study of $H$-unitary matrices arises from the study of indefinite inner product spaces. To see the connection, let $\langle \cdot, \cdot \rangle$ be the usual inner product, i.e. $\langle x, y \rangle = y^* x$ for $x, y \in \mathbb{C}^n$. An indefinite inner product in $\mathbb{C}^n$ is defined by

$$[x, y] = \langle Hx, y \rangle \quad \text{for any} \quad x, y \in \mathbb{C}^n.$$

Then $A \in M_n$ is *$H$-Hermitian* if $[Ax, y] = [x, Ay]$ for all $x, y \in \mathbb{C}^n$, equivalently, $HA = A^*H$; $U$ is $H$-unitary if $[x, y] = [Ux, Uy]$ for all $x, y \in \mathbb{C}^n$, equivalently, $H^{-1}U^*HU = I_n$. We refer the readers to [2, 5] for general background of indefinite inner product spaces.

The purpose of this paper is to characterize linear transformations sending $H_1$-unitary matrices in $M_n$ to $H_2$-unitary matrices in $M_m$ for two given invertible Hermitian matrices $H_1 \in M_n$ and $H_2 \in M_m$. Denote by $X \otimes Y$ the matrix $(x_{ij}Y)$ for two matrices $X = (x_{ij})$ and $Y$. We have the following.

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**Theorem 1** *Let $H_1 \in M_n$ and $H_2 \in M_m$ be invertible Hermitian matrices. A linear transformation $\phi : M_n \to M_m$ satisfies $\phi(\mathbf{U}(H_1)) \subseteq \mathbf{U}(H_2)$ if and only if there exist invertible matrices $S \in M_m$, $U, V \in \mathbf{U}(H_2)$ such that*

$$S^* H_2 S = [(I_a \oplus -I_b) \otimes H_1] \oplus [(I_c \oplus -I_d) \otimes (H_1^{-1})^t], \tag{1}$$

*and $\phi$ has the form*

$$A \mapsto US[(I_{a+b} \otimes A) \oplus (I_{c+d} \otimes A^t)]S^{-1}V, \tag{2}$$

*where $a$, $b$, $c$ and $d$ are nonnegative integers satisfying $(a+b+c+d)n = m$.*

Given two invertible Hermitian matrices $H_1 \in M_n$ and $H_2 \in M_m$, there does not always exist an invertible $S \in M_m$ satisfying (1). In such case, there will not be a linear transformation $\phi : M_n \to M_m$ such that $\phi(\mathbf{U}(H_1)) \subseteq \mathbf{U}(H_2)$. The next result show that the existence of an invertible $S \in M_m$ satisfying (1) is equivalent to the existence of a linear transformation $\phi : M_n \to M_m$ such that $\phi(\mathbf{U}(H_1)) \subseteq \mathbf{U}(H_2)$. Moreover, these conditions can be easily determined by the inertias of the matrices $H_1$ and $H_2$. (We say that the inertia of $H_i$ is $(p, q)$ if $H_i$ has $p$ positive eigenvalues and $q$ negative eigenvalues.)

**Theorem 2** *Let $H_1 \in M_n$ and $H_2 \in M_m$ be invertible Hermitian matrices such that $H_1$ has inertia $(p, q)$ and $H_2$ has inertia $(r, s)$. The following conditions are equivalent.*

(a) *There exists a linear transformation $\phi : M_n \to M_m$ such that $\phi(\mathbf{U}(H_1)) \subseteq \mathbf{U}(H_2)$.*

(b) *There exists an invertible matrix $S \in M_m$ satisfying* (1).

(c) *There are nonnegative integers $u$ and $v$ such that $(r, s) = u(p, q) + v(q, p)$.*

(d) *Either* (i) $p - q = r - s = 0$ *and* $(u + v)p = r$, *or*
(ii) $p \neq q$ *and* $(u, v) = (pr - qs, ps - qr)/(p^2 - q^2)$ *is a pair of nonnegative integers.*

**Example 3** Suppose $(m, n) = (6, 3)$. If $(r, s) = (5, 1)$, and $(p, q) = (2, 1)$, then there does not exists $(u, v)$ such that $(r, s) = u(p, q) + v(q, p)$. If we change $(r, s)$ to $(4, 2)$, then $(u, v) = (2, 0)$ is the unique solution for the equation $(r, s) = u(p, q) + v(q, p)$.

When $(H_1, H_2) = (I_n, I_m)$, our results reduce to the following theorem in [4].

**Corollary 4** *There is a linear transformation $\phi : M_n \to M_m$ such that $\phi(\mathbf{U}(I_n)) \subseteq \mathbf{U}(I_m)$ if and only if $m$ is a multiple of $n$, and there exist $U, V \in \mathbf{U}(I_m)$ such that $\phi$ has the form $A \mapsto U[(I_u \otimes A) \oplus (I_v \otimes A^t)]V$, where $u$ and $v$ are nonnegative integers satisfying $(u+v)n = m$.*

When $H_1 = H_2 = I_n$, our results reduce to that of Marcus [7], see also [3, 6].

**Corollary 5** *A linear transformation $\phi : M_n \to M_n$ satisfies $\phi(\mathbf{U}(I_n)) \subseteq \mathbf{U}(I_n)$ if and only if there exist $U, V \in \mathbf{U}(I_n)$ such that $\phi$ has the form $A \mapsto UAV$ or $A \mapsto UA^tV$.*

## 2 Auxiliary Results and Proofs

**Proof of Theorem 1.**

Let $J_{p,q} = I_p \oplus -I_q$ for any nonnegative integers $p$ and $q$, and let $\{E_{ij} : 1 \le i, j \le n\}$ be the standard basis of $M_n$.

Consider the ($\Leftarrow$) part. Note that $A \in \mathbf{U}(H_1)$ if and only if $A^t \in \mathbf{U}((H_1^{-1})^t)$. Let $u = a + b$ and $v = c + d$. If $A \in \mathbf{U}(H_1)$, then

$$\begin{aligned} & S^*(U^{-1}\phi(A)V^{-1})^* H_2 (U^{-1}\phi(A)V^{-1})S \\ = \ & [(I_u \otimes A^*) \oplus (I_v \otimes (A^t)^*)](S^*H_2S)[(I_u \otimes A) \oplus (I_v \otimes A^t)] \\ = \ & [(I_u \otimes A^*) \oplus (I_v \otimes (A^t)^*)][(J_{a,b} \otimes H_1) \oplus (J_{c,d} \otimes (H_1^{-1})^t)][(I_u \otimes A) \oplus (I_v \otimes A^t)] \\ = \ & [(J_{a,b} \otimes A^*H_1A) \oplus (J_{c,d} \otimes (A^t)^*(H_1^{-1})^t A^t)] \\ = \ & [(J_{a,b} \otimes H_1) \oplus (J_{c,d} \otimes (H_1^{-1})^t)] \\ = \ & S^*H_2S. \end{aligned}$$

Thus, $U^{-1}\phi(A)V^{-1} \in \mathbf{U}(H_2)$ and hence $\phi(A) \in \mathbf{U}(H_2)$ as well.

Next, consider the ($\Rightarrow$) part. Assume that $\phi : M_n \to M_m$ is a linear map satisfying $\phi(\mathbf{U}(H_1)) \subseteq \mathbf{U}(H_2)$. We will establish a sequence of assertions, which allow us to impose extra conditions on the transformation $\phi$, after we replace $\phi$ by a mapping of the form

$$A \mapsto V\phi(UAU^{-1})W, \quad U \in U(H_1), V, W \in U(H_2) \tag{3}$$

where $W = V^{-1}$ for all assertions except the first one. We always assume the extra condition once the triggering assertion is proved.

**Assertion 1** *Replacing $\phi$ by the mapping $A \mapsto \phi(I_n)^{-1}\phi(A)$, we may assume $\phi(I_n) = I_m$.*

**Assertion 2** *We may assume that $H_1 = J_{p,q}$ and $H_2 = J_{r,s}$ with $p \ge q$ and $r \ge s$.*

*Proof.* Let $S_1 \in M_n$ be invertible such that $S_1^* H_1 S_1 = J_{p,q}$ for some nonnegative integers $p$ and $q$ satisfying $p + q = n$. We may assume that $p \ge q$ because $\mathbf{U}(H_1) = \mathbf{U}(-H_1)$. Then $X \in M_n$ is $H_1$-unitary if and only if $S_1^{-1}XS_1$ is $J_{p,q}$-unitary. Similarly, there is an invertible $S_2$ such that $Y \in M_m$ is $H_2$-unitary if and only if $S_2^{-1}YS_2$ is $J_{r,s}$-unitary. Note that a linear map $\phi$ satisfies $\phi(\mathbf{U}(H_1)) \subseteq \mathbf{U}(H_2)$ if and only if the mapping $\psi$ defined by $A \mapsto S_2^{-1}\phi(S_1AS_1^{-1})S_2$ satisfies $\psi(\mathbf{U}(J_{p,q})) \subseteq \mathbf{U}(J_{r,s})$. Furthermore, $\phi$ has the asserted form if and only if $\psi$ has the same form. ∎

**Assertion 3** *The linear map $\phi$ sends $J_{p,q}$-Hermitian matrices to $J_{r,s}$-Hermitian matrices.*

*Proof.* Suppose $A$ is $J_{p,q}$-Hermitian. Then for any $t \in \mathbb{R}$,

$$J_{p,q}(e^{itA})^* J_{p,q} e^{itA} = e^{-itJ_{p,q}A^*J_{p,q}} e^{itA} = e^{-itA}e^{itA} = I_n.$$

So, $e^{itA} \in \mathbf{U}(J_{p,q})$. Now,

$$\phi(e^{itA}) = I_m + it\phi(A) - t^2\phi(A^2)/2! + \cdots$$

is $J_{r,s}$-unitary, i.e.,

$$I_m = J_{r,s}\phi(e^{itA})^* J_{r,s}\phi(e^{itA}) = I_m + it(\phi(A) - J_{r,s}\phi(A)^*J_{r,s}) + \cdots .$$

Thus, $\phi(A) - J_{r,s}\phi(A)^*J_{r,s} = 0$, i.e., $\phi(A)$ is $J_{r,s}$-Hermitian. ∎

**Assertion 4** *Suppose*

$$B = \begin{pmatrix} B_{11} & B_{12} \\ B_{21} & B_{22} \end{pmatrix} \in M_m$$

*is $J_{r,s}$-unitary and $J_{r,s}$-Hermitian, where $B_{11} \in M_r$ and $B_{22} \in M_s$. Then there exists a unitary matrix $X = X_1 \oplus X_2 \in M_m$ with $X_1 \in M_r$ and $X_2 \in M_s$ such that*

$$X^*BX = \begin{pmatrix} Z\sqrt{I_k + D^2} & 0 & D & 0 \\ 0 & Z_1 & 0 & 0 \\ -D & 0 & -Z\sqrt{I_k + D^2} & 0 \\ 0 & 0 & 0 & Z_2 \end{pmatrix}, \tag{4}$$

*where $Z, D \in M_k$, $Z_1 \in M_{r-k}$ and $Z_2 \in M_{s-k}$ such that $D$ is a diagonal matrix with positive diagonal entries and $Z$, $Z_1$ and $Z_2$ are diagonal matrices with diagonal entries in $\{1, -1\}$. Consequently, there is $S \in \mathbf{U}(J_{r,s})$ such that $S^{-1}BS$ is a diagonal matrix with diagonal entries in $\{1, -1\}$.*

*Proof.* Since $B$ is $J_{r,s}$-Hermitian, we have $B_{11} = B_{11}^*$, $B_{22} = B_{22}^*$ and $-B_{21} = B_{12}^*$. Let $U_1 \in M_r$ and $U_2 \in M_s$ be unitary such that

$$R = U_1^* B_{12} U_2 = \begin{pmatrix} D & 0_{k,s-k} \\ 0_{r-k,k} & 0_{r-k,s-k} \end{pmatrix},$$

where $D \in M_k$ is a diagonal matrix with positive diagonal entries arranged in descending order. Set $U = U_1 \oplus U_2$ and

$$\tilde{B} = J_{r,s}U^*BU = \begin{pmatrix} P & R \\ R^* & Q \end{pmatrix}.$$

Then $P = P^*$, $Q = Q^*$, $\tilde{B} \in \mathbf{U}(J_{r,s})$. So, $P^2 = I_r + RR^*$, $Q^2 = I_s + R^*R$ and $PR = RQ$. Hence, $P = P_1\sqrt{I_k + D^2} \oplus P_2$ and $Q = Q_1\sqrt{I_k + D^2} \oplus Q_2$, where $P_1, Q_1 \in M_k$ are unitary such that $P_1^2 = Q_1^2 = I_k$, $P_2 \in M_{r-k}$ and $Q_2 \in M_{s-k}$ satisfy $P_2^2 = I_{r-k}$, $Q_2^2 = I_{s-k}$, and

$\{P_1, Q_1, D, \sqrt{I_k + D^2}\}$ is a commuting family. Since $PR = RQ$, we see that $P_1 = Q_1$. Thus, there exist unitary matrices $W_0 \in M_k$, $W_1 \in M_{r-k}$ and $W_2 \in M_{s-k}$ such that $W_0 D = D W_0$ and all the matrices $W_1^* P_2 W_1$, $W_2^* Q_2 W_2$, and $W_0^* P_1 W_0 = W_0^* Q_1 W_0$ are in diagonal forms. Let $W = W_0 \oplus W_1 \oplus W_0 \oplus W_2$ and $X = UW$. Then $X^*BX$ has the asserted form.

Now, $X^*BX$ is permutationally similar to a direct sum of $Z_1 = W_1^* P_2 W_1$, $Z_2 = W_2^* Q_2 W_2$ and $2 \times 2$ matrices of the form

$$\text{(i)} \quad C = \begin{pmatrix} \sqrt{1+d^2} & d \\ -d & -\sqrt{1+d^2} \end{pmatrix} = \tilde{S}^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tilde{S}$$

or

$$\text{(ii)} \quad C = \begin{pmatrix} -\sqrt{1+d^2} & d \\ -d & \sqrt{1+d^2} \end{pmatrix} = \tilde{S}^{-1} \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tilde{S},$$

where

$$\tilde{S} = \begin{pmatrix} \sqrt{1+s^2} & s \\ s & \sqrt{1+s^2} \end{pmatrix} \quad \text{with} \quad s = \begin{cases} \{[\sqrt{1+d^2} - 1]/2\}^{1/2} & \text{if (i) holds,} \\ -\{[\sqrt{1+d^2} - 1]/2\}^{1/2} & \text{if (ii) holds.} \end{cases}$$

■

**Assertion 5** *Replacing $\phi$ by a mapping $A \mapsto S\phi(A)S^{-1}$ for some $S \in \mathbf{U}(J_{r,s})$, we may assume that for any diagonal matrix $D = \mathrm{diag}(d_1, \ldots, d_n)$,*

$$\phi(D) = d_1 I_{k_1} \oplus \cdots \oplus d_n I_{k_n} \oplus d_1 I_{k_{n+1}} \oplus \cdots \oplus d_n I_{k_{2n}},$$

*where $k_1, \ldots, k_{2n}$ are nonnegative integers with $k_1 + \cdots + k_n = r$ and $k_{n+1} + \cdots + k_{2n} = s$.*

*Proof.* To get the desired conclusion, it suffices to show that there is $S \in \mathbf{U}(J_{r,s})$ such that the mapping defined by $A \mapsto S\phi(A)S^{-1}$ satisfies

$$I_t \oplus 0_{n-t} \mapsto I_{k_1+\cdots+k_t} \oplus 0_{k_{t+1}+\cdots+k_n} \oplus I_{k_{n+1}+\cdots+k_{n+t}} \oplus 0_{k_{n+t+1}+\cdots+k_{2n}} \tag{5}$$

for any $1 \le t \le n$. We prove this claim by induction on $t$. For $t = 1$, let $A = J_{1,n-1}$. Since $A$ is $J_{p,q}$-Hermitian and $J_{p,q}$-unitary, by Assertions 3 and 4, $\phi(A) = S_1^{-1} D_1 S_1$, where $S_1 \in \mathbf{U}(J_{r,s})$ and $D_1 = I_{k_1} \oplus -I_{r-k_1} \oplus I_{k_{n+1}} \oplus -I_{s-k_{n+1}}$ for some nonnegative integers $k_1$ and $k_{n+1}$. Replacing $\phi$ by the mapping $X \mapsto S_1\phi(X)S^{-1}$, we have $\phi(J_{1,n-1}) = D_1$. Since $\phi(I_n) = I_m$, we see that $\phi([1] \oplus 0_{n-1})$ has the asserted form.

Now, we assume that (5) holds for $t-1$. Let $A = J_{t,n-t}$ and $K = I_{t-1,n-t+1}$. By induction assumption and the fact that $\phi(I_n) = I_m$, we may assume that $L = \phi(K) = I_a \oplus -I_{r-a} \oplus I_b \oplus -I_{s-b}$, where $a = k_1 + \cdots + k_{t-1}$ and $b = k_{n+1} + \cdots + k_{n+t}$. Let $B = \phi(A)$. Since $(A \pm iK)/\sqrt{2} \in \mathbf{U}(J_{p,q})$, it follows that $(B \pm iL)/\sqrt{2} \in \mathbf{U}(J_{r,s})$. So,

$$2I_m = J_{r,s}(B \pm iL)^* J_{r,s}(B \pm iL) = (B \mp iL)(B \pm iL) = B^2 + L^2 \pm i(BL - LB).$$

Thus $BL = LB$, i.e., $LBL = B$. Since $L = I_a \oplus -I_{r-a} \oplus I_b \oplus -I_{s-b}$, $B$ has the form

$$\begin{pmatrix} B_{11} & 0 & B_{13} & 0 \\ 0 & B_{22} & 0 & B_{24} \\ B_{31} & 0 & B_{33} & 0 \\ 0 & B_{42} & 0 & B_{44} \end{pmatrix},$$

according to the block structure of $L$. On the other hand, both $A$ and $A - 2[I_{t-1} \oplus 0_{n-t+1}]$ are $J_{p,q}$-unitary. Thus, $B$ and $\tilde{B} = B - 2(I_a \oplus 0_{r-a} \oplus I_b \oplus 0_{s-b})$ are $J_{r,s}$-unitary, i.e., $B^* J_{r,s} B J_{r,s} = I_m = \tilde{B}^* J_{r,s} \tilde{B} J_{r,s}$. It follows that $B_{11} = I_a$, $B_{33} = I_b$, $B_{13}$ and $B_{31}$ are zero, and $\begin{pmatrix} B_{22} & B_{24} \\ B_{42} & B_{44} \end{pmatrix} \in \mathbf{U}(J_{r-a,s-b})$. By Assertion 4, there exists

$$\begin{pmatrix} S_{22} & S_{24} \\ S_{42} & S_{44} \end{pmatrix} \in \mathbf{U}(J_{r-a,s-b}) \quad \text{such that} \quad S_t = \begin{pmatrix} I_a & 0 & 0 & 0 \\ 0 & S_{22} & 0 & S_{24} \\ 0 & 0 & I_b & 0 \\ 0 & S_{42} & 0 & S_{44} \end{pmatrix} \in \mathbf{U}(J_{r,s})$$

and

$$S_t^{-1} \phi(A) S_t = S_t^{-1} B S_t = I_a \oplus I_{k_t} \oplus -I_{r-a-k_t} \oplus I_b \oplus I_{k_{n+t}} \oplus -I_{s-b-k_{n+t}}.$$

Now, we can replace $\phi$ by $X \mapsto S_t^{-1} \phi(X) S_t$ and assume that $\phi(I_t \oplus 0_{n-t})$ has the desired form. ∎

We need some more notations and definitions in the rest of our proof. We have to consider different cases according to the following three types of ordered pair $(u, v)$ of integers with $1 \le u < v \le n$:

$$\text{I}: \ 1 \le u \le p < v \le n; \qquad \text{II.a}: \ 1 \le u < v \le p; \qquad \text{II.b}: \ p < u < v \le n.$$

For any $B \in M_n$ and $C \in M_2$, let

$$A = B(C; [u, v])$$

be the matrix in $M_n$ obtained from $B$ by replacing $\begin{pmatrix} b_{uu} & b_{uv} \\ b_{vu} & b_{vv} \end{pmatrix}$ by $C$. Similarly, for any $C \in M_k$, and $B = (B_{ij}) \in M_m$, where $B_{ij} \in M_{k_i \times k_j}$ and $k = k_u + k_v + k_{n+u} + k_{n+v}$, let

$$A = B(C; \{u, v\})$$

be the matrix in $M_m$ obtained from $B$ by replacing its submatrix

$$\begin{pmatrix} B_{uu} & B_{uv} & B_{u(n+u)} & B_{u(n+v)} \\ B_{vu} & B_{vv} & B_{v(n+u)} & B_{v(n+v)} \\ B_{(n+u)u} & B_{(n+u)v} & B_{(n+u)(n+u)} & B_{(n+u)(n+v)} \\ B_{(u+v)u} & B_{(n+v)v} & B_{(n+v)(n+u)} & B_{(n+v)(n+v)} \end{pmatrix}$$

by $C$. For any $1 \le u < v < w \le n$, we define $B(C; [u, v, w])$ and $B(C; \{u, v, w\})$ in a similar way. Furthermore, we need the matrices in the following table in our proofs.

**Table**

| Type | I | II.a | II.b |
|---|---|---|---|
| $M_B =$ | $\begin{pmatrix} \sqrt{2} & \pm i \\ \pm i & -\sqrt{2} \end{pmatrix}$ | $\begin{pmatrix} \frac{1}{\sqrt{2}} & \pm\frac{1}{\sqrt{2}} \\ \pm\frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$ | same as Type II.a |
| $M_C =$ | $\begin{pmatrix} \sqrt{2} & \pm 1 \\ \mp 1 & -\sqrt{2} \end{pmatrix}$ | $\begin{pmatrix} \frac{1}{\sqrt{2}} & \pm\frac{i}{\sqrt{2}} \\ \mp\frac{i}{\sqrt{2}} & \frac{-1}{\sqrt{2}} \end{pmatrix}$ | same as Type II.a |
| $M_+ =$ | $\begin{pmatrix} \sqrt{2} & \pm\frac{i-1}{\sqrt{2}} \\ \pm\frac{i+1}{\sqrt{2}} & -\sqrt{2} \end{pmatrix}$ | $\begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1+i}{2} \\ \frac{1-i}{2} & \frac{-1}{\sqrt{2}} \end{pmatrix}$ | same as Type II.a |
| $J =$ | $J_{2\alpha,\alpha} \oplus J_{2\beta,\beta}$ | $I_{3\alpha} \oplus I_{3\beta}$ | $J_{\alpha,2\alpha} \oplus J_{\beta,2\beta}$ |
| $X =$ | $\begin{pmatrix} \frac{\sqrt{3}+1}{2} & \frac{\sqrt{3}-1}{2} & 1 \\ \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} & 1 \\ 1 & 1 & \sqrt{3} \end{pmatrix}$ | $\begin{pmatrix} \frac{\sqrt{3}-1}{2\sqrt{3}} & \frac{-\sqrt{3}-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{-\sqrt{3}-1}{2\sqrt{3}} & \frac{\sqrt{3}-1}{2\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{pmatrix}$ | $\begin{pmatrix} \sqrt{3} & 1 & 1 \\ 1 & \frac{\sqrt{3}+1}{2} & \frac{\sqrt{3}-1}{2} \\ 1 & \frac{\sqrt{3}-1}{2} & \frac{\sqrt{3}+1}{2} \end{pmatrix}$ |

**Assertion 6** *Suppose $k_1, \ldots, k_{2n}$ have the meaning in Assertion 5. For any $1 \le u < v \le n$, let $B = \phi(E_{uv} + E_{vu})$ and $C = \phi(E_{uv} - E_{vu})$. The following conclusions hold.*

1. *If $(u, v)$ is of Type I, then $k_u = k_{n+v}$ and $k_{n+u} = k_v$. Moreover, for $K = iB$, $C$ or $(iB - C)/\sqrt{2}$, $K = 0_m(R; \{u, v\})$, where*

$$R = \begin{pmatrix} 0 & R_1 & 0 & R_2 \\ R_1^* & 0 & R_3 & 0 \\ 0 & -R_3^* & 0 & R_4 \\ -R_2^* & 0 & R_4^* & 0 \end{pmatrix} \tag{6}$$

   *and $R^2 = -I_k$.*

2. *If $(u, v)$ is of Type II.a/II.b, then $k_u = k_v$ and $k_{n+u} = k_{n+v}$. Moreover, for $K = B$, $iC$ or $(B + iC)/\sqrt{2}$, $K = 0_m(R; \{u, v\})$, where $R$ has the form (6) and $R^2 = I_k$.*

*Consequently,*

$$k_1 = \cdots = k_p = k_{n+p+1} = \cdots = k_{2n} = \alpha \quad \text{and} \quad k_{p+1} = \cdots = k_n = k_{n+1} = \cdots = k_{n+p} = \beta.$$

*Proof.* Suppose $(u, v)$ is of Type I. Let $A = I_n(M; [u, v])$ with $M = M_B$, $M_C$ or $M_+$, where $M_B$, $M_C$ and $M_+$ are the matrices of Type I in the Table. Then $A$ is both $J_{p,q}$-unitary and $J_{p,q}$-Hermitian. Write $A = (a_{ij})$ and $A_d = \operatorname{diag}(a_{11}, \ldots, a_{nn})$. By Assertion 5, $\phi(A_d) = D = I_m(\tilde{D}; \{u, v\})$ with

$$\tilde{D} = a_{uu} I_{k_u} \oplus a_{vv} I_{k_v} \oplus a_{uu} I_{k_{n+u}} \oplus a_{vv} I_{k_{n+v}}. \tag{7}$$

Moreover, $\phi(A) = D \pm |a_{uv}| K = D \pm K$ are $J_{r,s}$-Hermitian as well as $J_{r,s}$-unitary with $K = iB$, $C$ or $(iB - C)/\sqrt{2}$ depending on $M = M_B$, $M_C$ or $M_+$. Hence,

$$I_m = J_{r,s}(D \pm K)^* J_{r,s}(D \pm K) = (D \pm K)^2 = D^2 + K^2 \pm (DK + KD).$$

Thus $DK + KD = 0$; by the block structure of $D = I_m(\tilde{D}; \{u, v\})$, only the following eight blocks

$$K_{uv},\ K_{u(n+v)},\ K_{vu},\ K_{v(n+u)},\ K_{(n+u)v},\ K_{(n+u)(n+v)},\ K_{(n+v)u},\ K_{(n+v)(n+u)},$$

can be nonzero. As $K$ is $J_{r,s}$-Hermitian,

$$K_{vu} = K_{uv}^*, \quad K_{(n+v)u} = -K_{u(n+v)}^*, \quad K_{(n+u)v} = -K_{v(n+u)}^*, \quad K_{(n+v)(n+u)} = K_{(n+u)(n+v)}^*.$$

Hence, $K = 0_m(R; \{u, v\})$, where $R$ has the form (6). Since $I_m = D^2 + K^2$, it follows that $I_k = \tilde{D}^2 + R^2$, where $\tilde{D}$ is the matrix in (7) and $k = k_u + k_v + k_{n+u} + k_{n+v}$. Since $\tilde{D}^2 = 2I_k$, we have $R^2 = -I_k$.

Suppose $(u, v)$ is of Type II.a/II.b. Let $A = I_n(M; [u, v]) \in \mathbf{U}(J_{p,q})$ with $M = M_B$, $M_C$ or $M_+$ of Type II.a/II.b in the Table. Then $\phi(A) = D \pm (K/\sqrt{2})$ with $D = \phi(\operatorname{diag}(a_{11}, \ldots, a_{nn}))$ and $K = iB$, $C$ or $(B + iC)/\sqrt{2}$. One can carry out a similar analysis as in the Type I case and conclude that $K = 0_m(R; \{u, v\})$, where $R$ has the form (6) with $R^2 = I_k$. ∎

In the following, we always assume that $\alpha, \beta, k_1, \ldots, k_{2n}$ have the meaning in Assertions 5 and 6. We show that additional assumptions can be imposed on the matrices $\phi(E_{uv} + E_{vu})$ and $\phi(E_{uv} - E_{vu})$ for $1 \leq u < v \leq n$. However, it is inconvenient to use the block forms arising from Assertions 5 and 6. Instead, we introduce the following block permutation matrices:

$$P = \begin{pmatrix} I_{\alpha p} & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{\alpha q} \\ 0 & 0 & I_{\beta p} & 0 \\ 0 & I_{\beta q} & 0 & 0 \end{pmatrix}, \quad Q_1 = \begin{pmatrix} I_\alpha & 0 & 0 & 0 \\ 0 & 0 & 0 & I_\alpha \\ 0 & 0 & I_\beta & 0 \\ 0 & I_\beta & 0 & 0 \end{pmatrix} \quad \text{and} \quad Q_2 = \begin{pmatrix} 0 & 0 & I_\alpha & 0 \\ 0 & 0 & 0 & I_\alpha \\ I_\beta & 0 & 0 & 0 \\ 0 & I_\beta & 0 & 0 \end{pmatrix},$$

where $P \in M_m$ and $Q_1, Q_2 \in M_{2(\alpha+\beta)}$. Then, for $D = \operatorname{diag}(d_1, \ldots, d_n) \in M_n$ in Assertion 5, we have $P^* \phi(D) P = (D \otimes I_\alpha) \oplus (D \otimes I_\beta)$. Moreover, with the matrix $R$ in Assertion 6, we have

$$P^*[0_m(R; \{u, v\})]P = \begin{cases} 0_m(Q_1^* R Q_1; \{u, v\}) & \text{if } (u, v) \text{ is of Type I,} \\ 0_m(R; \{u, v\}) & \text{if } (u, v) \text{ is of Type II.a,} \\ 0_m(Q_2^* R Q_2; \{u, v\}) & \text{if } (u, v) \text{ is of Type II.b,} \end{cases}$$

where $Q_2^* R Q_2$ also has the form (6) after relabeling the blocks $R_1, \ldots, R_4$, and $Q_1^* R Q_1$ has the form

$$T = \begin{pmatrix} 0 & T_1 & 0 & T_2 \\ -T_1^* & 0 & T_3 & 0 \\ 0 & T_3^* & 0 & T_4 \\ T_2^* & 0 & -T_4^* & 0 \end{pmatrix}. \tag{8}$$

Furthermore, it is clear that $R$ has the form (6) if $iR$ has the form (8). Thus, Assertion 6 can be rewritten as follows.

**Assertion 7** *For any $1 \leq u < v \leq n$, let $B = \phi(E_{uv} + E_{vu})$ and $C = \phi(E_{uv} - E_{vu})$. Then*

$$P^*BP = 0_m(R; \{u, v\}), \ \ P^*CP = 0_m(T; \{u, v\}), \ \ \text{and } P^*[(B + iC)/\sqrt{2}]P = 0_m(\tilde{R}; \{u, v\}),$$

*where $R$ and $\tilde{R}$ have the form (6) and $T$ has the form (8) such that $R^2 = \tilde{R}^2 = I_k = -T^2$.*

**Assertion 8** *Replacing $\phi$ by the mapping $A \mapsto S\phi(A)S^{-1}$ for some $S \in \mathbf{U}(J_{r,s})$, we may assume that for any $A = E_{uv} + E_{vu}$ with $1 \leq u < v \leq n$,*

$$P^*\phi(A)P = (A \otimes I_\alpha) \oplus (A \otimes I_\beta).$$

*Proof.* Let

$$F = \begin{pmatrix} 0 & I_\alpha \\ I_\alpha & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & I_\beta \\ I_\beta & 0 \end{pmatrix} \in M_{2(\alpha+\beta)}. \tag{9}$$

By Assertion 7, $P^*\phi(E_{uv} + E_{vu})P = 0_m(R; \{u, v\})$, where $R$ has the form (6). Since $R^2 = I_k$, it follows that

$$\begin{pmatrix} R_1 & R_2 \\ -R_3^* & R_4 \end{pmatrix} \in \mathbf{U}(J_{\alpha,\beta}) \quad \text{and} \quad U = \begin{pmatrix} I_\alpha & 0 & 0 & 0 \\ 0 & R_1 & 0 & R_2 \\ 0 & 0 & I_\beta & 0 \\ 0 & -R_3^* & 0 & R_4 \end{pmatrix} \in \mathbf{U}(J_{2\alpha,2\beta}).$$

Moreover, $R = U^{-1}FU$. Set $S_{uv} = P[I_m(U; \{u, v\})]P^*$. Then $S_{uv}$ is $J_{r,s}$-unitary and

$$\begin{aligned} S_{uv}\phi(E_{uv} + E_{vu})S_{uv}^{-1} &= P[I_m(U; \{u, v\})][0_m(R; \{u, v\})][I_m(U^{-1}; \{u, v\})]P^* \\ &= P[0_m(URU^{-1}; \{u, v\})]P^* \\ &= P[0_m(F; \{u, v\})]P^*. \end{aligned}$$

By the block structure of $S_{uv}$, one can see that for any $A \in M_m$, all blocks of $S_{uv}AS_{uv}^{-1}$ are the same as those of $A$ except for the blocks indexed by $v$ and $n + v$. Hence,

$$S_{1v}P[0_m(R; \{1, v'\})]P^*S_{1v}^{-1} = P[0_m(R; \{1, v'\})]P^* \quad \text{for all} \quad v' \neq v.$$

Replacing $\phi$ by the mapping $A \mapsto S\phi(A)S^{-1}$ with $S = S_{12} \cdots S_{1n}$, we may assume that the assertion holds for $(u, v) = (1, w)$ with $2 \le w \le n$. Note also that the conclusion of Assertion 5 is not affected.

It remains to show that the conclusion holds for other $(u, v)$ if $n > 2$. Let $A = I_n(X; [1, u, v])$, where $X = (x_{ij})$ is defined in the Table, depending on the type of $(u, v)$. Assertion 7 ensures that $P^*\phi(E_{uv} + E_{vu})P = I_m(R; \{u, v\})$, where $R$ has the form in (6). Together with Assertions 5 and 7, $P^*\phi(A)P = I_m(Y; \{1, u, v\})$, where

$$Y = \begin{pmatrix} x_{11}I_\alpha & x_{12}I_\alpha & x_{13}I_\alpha & 0 & 0 & 0 \\ x_{21}I_\alpha & x_{22}I_\alpha & x_{23}R_1 & 0 & 0 & x_{23}R_2 \\ x_{31}I_\alpha & x_{32}R_1^* & x_{33}I_\alpha & 0 & x_{32}R_3 & 0 \\ 0 & 0 & 0 & x_{11}I_\beta & x_{12}I_\beta & x_{13}I_\beta \\ 0 & 0 & -x_{23}R_3^* & x_{21}I_\beta & x_{22}I_\beta & x_{23}R_4 \\ 0 & -x_{32}R_2^* & 0 & x_{31}I_\beta & x_{32}R_4^* & x_{33}I_\beta \end{pmatrix}.$$

Since $\phi(A) \in \mathbf{U}(J_{r,s})$, it follows that $Y$ is $J$-unitary, where $J$ is defined in the Table depending on the type of $(u, v)$. Comparing the $(1,3)$-th, $(1,5)$-th, $(1,6)$-th and $(4,6)$-th blocks on both sides of the equation $Y^*JY = J$, we see that $R_1 = I_\alpha$, $R_3 = 0$, $R_2 = 0$ and $R_4 = I_\beta$, respectively. Hence $R = F$. ∎

The next two assertions deal with $\phi(E_{uv} - E_{vu})$.

**Assertion 9** *Replacing $\phi$ by the mapping $A \mapsto S\phi(A)S^{-1}$ for some $S \in \mathbf{U}(J_{r,s})$, we may further assume that*

$$P^*\phi(E_{12} - E_{21})P = [(E_{12} - E_{21}) \otimes J_{p_1,p_2}] \oplus [(E_{12} - E_{21}) \otimes J_{q_1,q_2}],$$

*where $p_1$, $p_2$, $q_1$ and $q_2$ are nonnegative integers satisfying $p_1 + p_2 = \alpha$ and $q_1 + q_2 = \beta$.*

*Proof.* Let

$$G = \begin{pmatrix} 0 & J_{p_1,p_2} \\ -J_{p_1,p_2} & 0 \end{pmatrix} \oplus \begin{pmatrix} 0 & J_{q_1,q_2} \\ -J_{q_1,q_2} & 0 \end{pmatrix} \in M_{2(\alpha+\beta)}. \tag{10}$$

Suppose $B = \phi(E_{12} + E_{21})$ and $C = \phi(E_{12} - E_{21})$. By assertion 7, we may write

$$P^*CP = 0_m(T; \{1, 2\}) \quad \text{and} \quad P^*(B + iC)P = \sqrt{2}[0_m(\tilde{R}; \{1, 2\})],$$

where $T$ and $\tilde{R}$ have the form (8) and (6) respectively, such that $\tilde{R}^2 = I_k = -T^2$. Since $P^*BP = 0_m(F; \{1, 2\})$, we have $F + iT = \sqrt{2}\tilde{R}$. With $F^2 = I_k$, we see that $FT + TF = 0$, i.e.,

$$\begin{pmatrix} 0 & -T_1^* & 0 & T_3 \\ T_1 & 0 & T_2 & 0 \\ 0 & T_2^* & 0 & -T_4^* \\ T_3^* & 0 & T_4 & 0 \end{pmatrix} = FTF = -T = -\begin{pmatrix} 0 & T_1 & 0 & T_2 \\ -T_1^* & 0 & T_3 & 0 \\ 0 & T_3^* & 0 & T_4 \\ T_2^* & 0 & -T_4^* & 0 \end{pmatrix}.$$

So, $T_2 = -T_3$ and both $T_1$ and $T_4$ are Hermitian.

Now there exist unitary matrices $U_1 \in M_\alpha$ and $U_2 \in M_\beta$ such that $\tilde{T}_2 = U_1 T_2 U_2^* = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$, where $D \in M_l$ is a diagonal matrix with positive entries in descending order, i.e., $D = d_1 I_{h_1} \oplus \cdots \oplus d_t I_{h_t}$ with $d_1 > \cdots > d_t$ and $h_1 + \cdots + h_t = l$. Let $U = U_1 \oplus U_1 \oplus U_2 \oplus U_2$. Then $UTU^*$ has the same block form as $T$. Let

$$\tilde{T} = UTU^* = \begin{pmatrix} 0 & \tilde{T}_1 & 0 & \tilde{T}_2 \\ -\tilde{T}_1 & 0 & -\tilde{T}_2 & 0 \\ 0 & -\tilde{T}_2^* & 0 & \tilde{T}_4 \\ \tilde{T}_2^* & 0 & -\tilde{T}_4 & 0 \end{pmatrix}.$$

Then $\tilde{T}_1^2 = I_\alpha + \tilde{T}_2 \tilde{T}_2^*$; so, $\tilde{T}_1$ has the form

$$\tilde{T}_1 = \sqrt{1 + d_1^2} X_1 \oplus \cdots \oplus \sqrt{1 + d_t^2} X_t \oplus X_{t+1},$$

where $X_i \in M_{h_i}$ and $X_{t+1} \in M_{\alpha - l}$ are unitary and Hermitian. Similarly,

$$\tilde{T}_4 = \sqrt{1 + d_1^2} Y_1 \oplus \cdots \oplus \sqrt{1 + d_t^2} Y_t \oplus Y_{t+1},$$

where $Y_i \in M_{h_i}$ and $Y_{t+1} \in M_{\beta - l}$ are unitary and Hermitian. Set $X = X_1 \oplus \cdots \oplus X_t$ and $Y = Y_1 \oplus \cdots \oplus Y_t$. Since $\tilde{T}_1 \tilde{T}_2 + \tilde{T}_2 \tilde{T}_4 = 0$, it follows that $X = -Y$. Then

$$\tilde{T}_1 = \sqrt{I_l + D^2} X \oplus X_{t+1}, \quad \tilde{T}_4 = \sqrt{I_l + D^2}(-X) \oplus Y_{t+1},$$

and $\{X, D, \sqrt{I_l + D^2}\}$ is a commuting family. Therefore, there exist unitary $V_0 \in M_l$, $V_1 \in M_{\alpha - l}$ and $V_2 \in M_{\beta - l}$ such that $Z = V_0 X V_0^*$, $Z_1 = V_1 X_{t+1} V_1^*$ and $Z_2 = V_2 Y_{t+1} V_2^*$ are diagonal matrices with diagonal entries in $\{1, -1\}$. Let $W_1 = (V_0 \oplus V_1) U_1$, $W_2 = (V_0 \oplus V_2) U_2$ and $W = W_1 \oplus W_1 \oplus W_2 \oplus W_2$. Then

$$WTW^* = \begin{pmatrix} 0 & D_1 & 0 & D_2 \\ -D_1 & 0 & -D_2 & 0 \\ 0 & -D_2 & 0 & -D_4 \\ D_2 & 0 & D_4 & 0 \end{pmatrix}$$

with $D_1 = (Z\sqrt{I_l + D^2}) \oplus Z_1$, $D_4 = (Z\sqrt{I_l + D^2}) \oplus Z_2$ and $D_2 = \begin{pmatrix} D & 0 \\ 0 & 0 \end{pmatrix}$. Let

$$L = \begin{pmatrix} Z[(\sqrt{I_l + D^2} - I_l)/2]^{1/2} & 0 \\ 0 & 0 \end{pmatrix}$$

and

$$K = \begin{pmatrix} \sqrt{I_l + LL^*} & 0 & L & 0 \\ 0 & \sqrt{I_l + LL^*} & 0 & L \\ L^* & 0 & \sqrt{I_l + L^*L} & 0 \\ 0 & L^* & 0 & \sqrt{I_l + L^*L} \end{pmatrix}.$$

Then $K \in \mathbf{U}(J_{2\alpha,2\beta})$, $K^{-1} = J_{2\alpha,2\beta} K J_{2\alpha,2\beta}$, and

$$WTW^* = K^{-1} \begin{pmatrix} 0 & Z \oplus Z_1 & 0 & 0 \\ -(Z \oplus Z_1) & 0 & 0 & 0 \\ 0 & 0 & 0 & -(Z \oplus Z_2) \\ 0 & 0 & Z \oplus Z_2 & 0 \end{pmatrix} K.$$

Now, there exist permutation matrices $Q_1$ and $Q_2$ such that $Q_1(Z \oplus Z_1)Q_1^* = J_{p_1,p_2}$ and $Q_2(Z \oplus Z_2)Q_2^* = J_{q_1,q_2}$ for some nonnegative integers $p_1$, $p_2$, $q_1$ and $q_2$ satisfying $p_1 + p_2 = \alpha$ and $q_1 + q_2 = \beta$. Let $Q = Q_1 \oplus Q_1 \oplus Q_2 \oplus Q_2$ and $S = QKW$. Then $STS^{-1} = G$.

Define $\tilde{W} = (I_n \otimes W_1) \oplus (I_n \otimes W_2)$, $\tilde{Q} = (I_n \otimes Q_1) \oplus (I_n \otimes Q_2)$,

$$\tilde{K} = \begin{pmatrix} I_n \otimes \sqrt{I_l + LL^*} & I_n \otimes L \\ I_n \otimes L^* & I_n \otimes \sqrt{I_l + L^*L} \end{pmatrix}$$

and $\tilde{S} = \tilde{Q}\tilde{K}\tilde{W}$. Then

$$\begin{aligned} (P\tilde{S}P^*)C(P\tilde{S}P^*)^{-1} &= P\tilde{S}[0_m(T;\{1,2\})]\tilde{S}^{-1}P^* \\ &= P[0_m(STS^{-1};\{1,2\})]P^* \\ &= P[0_m(G;\{u,v\})]P^*. \end{aligned}$$

On the other hand, we have $SFS^{-1} = F$. Hence, for any $1 \le u < v \le n$,

$$(P\tilde{S}P^*)\phi(E_{uv} + E_{vu})(P\tilde{S}P^*)^{-1} = P[0_m(SFS^{-1};\{u,v\})]P^* = P[0_m(F;\{u,v\})]P^*.$$

Since $P\tilde{Q}P^*$, $P\tilde{K}P^*$ and $P\tilde{W}P^*$ are all $J_{r,s}$-unitary, $P\tilde{S}P^* \in \mathbf{U}(J_{r,s})$. Replacing $\phi$ by $A \mapsto (P\tilde{S}P^*)\phi(A)(P\tilde{S}P^*)^{-1}$, we get the desired result. ∎

**Assertion 10** *For any $A = E_{uv} - E_{vu}$ with $1 \le u < v \le n$, we have*

$$P^*\phi(A)P = (A \otimes J_{p_1,p_2}) \oplus (A \oplus J_{q_1,q_2}),$$

*where $p_1, p_2, q_1, q_2$ are defined as in Assertion 9.*

*Proof.* Define $G$ as in (10). We first consider $(u,v) = (1,w)$. When $w = 2$, the result is valid by Assertion 9. Consider $w > 2$. By Assertion 7, $P^*\phi(E_{1w} - E_{w1})P = 0_m(T;\{1,w\})$, where $T$ has the form (8). Let $A = I_n((XD_1);[1,2,w])$, where $D_1 = \mathrm{diag}(-1,1,1)$ and $X = (x_{ij})$ is defined in the Table, depending on the type of $(2,w)$. By Assertions 8 and 9, $P^*\phi(A)P = I_m(Y;\{1,2,w\})$, where

$$Y = \begin{pmatrix} -x_{11}I_\alpha & x_{12}J_{p_1,p_2} & x_{13}T_1 & 0 & 0 & x_{13}T_2 \\ -x_{21}J_{p_1,p_2} & x_{22}I_\alpha & x_{23}I_\alpha & 0 & 0 & 0 \\ -x_{31}T_1^* & x_{32}I_\alpha & x_{13}I_\alpha & x_{13}T_3 & 0 & 0 \\ 0 & 0 & x_{13}T_3^* & -x_{11}I_\beta & x_{12}J_{q_1,q_2} & x_{13}T_4 \\ 0 & 0 & 0 & -x_{21}J_{q_1,q_2} & x_{22}I_\beta & x_{23}I_\beta \\ x_{13}T_2^* & 0 & 0 & -x_{31}T_4^* & x_{32}I_\beta & x_{33}I_\beta \end{pmatrix}.$$

Since $\phi(A) \in \mathbf{U}(J_{r,s})$, it follows that $Y$ is $J$-unitary, where $J$ is defined in the Table depending on the type of $(u, v)$. Comparing the $(2, 3)$-th, $(2, 4)$-th, $(2, 6)$-th and $(5, 6)$-th blocks on both sides of $Y^*JY = J$, we see that $T = G$.

For those $(u, v)$ with $1 < u < v \leq n$, we can apply the preceding analysis to the matrix $A = I_n((D_2X); [1, u, v])$ with $D_2 = \mathrm{diag}(1, 1, -1)$ to get the conclusion. ∎

By Assertions 5, 8–10, we see that for any $1 \leq i, j \leq n$,

$$\begin{aligned} P^*\phi(E_{ii})P &= [E_{ii} \otimes I_\alpha] \oplus [E_{ii} \otimes I_\beta], \\ P^*\phi(E_{ij} + E_{ji})P &= [(E_{ij} + E_{ji}) \otimes I_\alpha] \oplus [(E_{ij} + E_{ji}) \otimes I_\beta], \\ P^*\phi(E_{ij} - E_{ji})P &= [(E_{ij} - E_{ji}) \otimes J_{p_1,p_2}] \oplus [(E_{ij} - E_{ji}) \otimes J_{q_1,q_2}]. \end{aligned}$$

Thus, there exists a permutation matrix $Q \in M_m$ such that for $E = E_{ii}$, $(E_{ij} + E_{ji})$ and $(E_{ij} - E_{ji})$,

$$Q^*\phi(E)Q = (I_{p_1+q_1} \otimes E) \oplus (I_{p_2+q_2} \otimes E^t).$$

Set $u = p_1 + q_1$ and $v = p_2 + q_2$. Then for any $A \in M_n$, we have

$$\phi(A) = Q[(I_u \otimes A) \oplus (I_v \otimes A^t)]Q^*.$$

It remains to prove that $Q$ can be chosen to satisfy

$$Q^*J_{r,s}Q = (J_{a,b} \otimes J_{p,q}) \oplus (J_{c,d} \otimes J_{p,q})$$

for some nonnegative integers $a$, $b$, $c$ and $d$ satisfying $a + b = u$ and $c + d = v$. To this end, let

$$Q^*J_{r,s}Q = J_1 \oplus \cdots \oplus J_t,$$

where $t = m/n = u + v$ and $J_i \in M_n$ are diagonal matrices with diagonal entries in $\{1, -1\}$. Note that for any $A \in \mathbf{U}(J_{p,q})$, the matrix $Q^*\phi(A)Q$ is $Q^*J_{r,s}Q$-unitary. It follows that $A$ is $J_i$-unitary for all $i$. Thus, $\mathbf{U}(J_{p,q}) \subseteq \mathbf{U}(J_i)$, and hence $J_i = \pm J_{p,q}$. Now, we may further permute the blocks $J_i$ and assume that $Q$ satisfies $Q^*J_\gamma Q = (J_{a,b} \otimes J_{p,q}) \oplus (J_{c,d} \otimes J_{p,q})$ with $a + b = u$ and $c + d = v$. The proof of Theorem 1 is complete. ∎

**Proof of Theorem 2.**

The equivalence of (c) and (d) can be verified readily.

(a) $\Rightarrow$ (c): Suppose there is a linear transformation $\phi : M_n \to M_m$ such that $\phi(\mathbf{U}(H_1)) \subseteq \mathbf{U}(H_2)$. By Theorem 1, we have $(a + b + c + d)n = m$, i.e., $m$ is a multiple of $n$. Moreover, comparing the inertias of the matrices on both side of (1), we see that $(r, s) = u(p, q)+v(q, p)$, where $u = a + c$ and $v = b + d$.

(c) $\Rightarrow$ (b): Suppose there are nonnegative integers $u, v$ such that $(r, s) = u(p, q) + v(q, p)$. Then $r + s = (u + v)(p + q)$, hence $m$ is a multiple of $n$. Also, $H_2$ and $(I_u \oplus -I_v) \otimes H_1$ will have the same inertia. Thus, there exists $S \in M_m$ satisfying (1) with $(a, b, c, d) = (u, v, 0, 0)$.

(b) $\Rightarrow$ (a): Suppose there exists $S \in M_m$ satisfying (1). We can construct $\phi$ of the form (2) with $U = V = I_m$ so that $\phi(\mathbf{U}(H_1)) \subseteq \mathbf{U}(H_2)$. ∎

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