Linear maps transforming the unitary group

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Abstract
Let \( U(n) \) be the group of \( n \times n \) unitary matrices. We show that if \( \phi \) is a linear transformation sending \( U(n) \) into \( U(m) \), then \( m \) is a multiple of \( n \), and \( \phi \) has the form
\[
A \mapsto V[(A \otimes I_s) \oplus (A^t \otimes I_r)]W
\]
for some \( V, W \in U(m) \). From this result, one easily deduces the characterization of linear operators that map \( U(n) \) into itself obtained by Marcus. Further generalization of the main theorem is also discussed.

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1 Main Result

Denote by \( M_n \) the algebra of \( n \times n \) complex matrices. Let \( U(n) \) be the group of \( n \times n \) unitary matrices. The purpose of this note is to prove the following result.

**Theorem 1** Suppose \( \phi : M_n \to M_m \) is a linear transformation satisfying \( \phi(U(n)) \subseteq U(m) \). Then \( m \) is a multiple of \( n \) and
\[
\phi(A) = V[(A \otimes I_s) \oplus (A^t \otimes I_r)]W
\]
for some fixed \( V, W \in U(m) \).

For any linear map \( \phi : M_n \to M_m \) satisfying \( \phi(U(n)) \subseteq U(m) \), one can replace it by the mapping \( \psi \) of the form \( A \mapsto \phi(I_n)^{-1}\phi(A) \). Then \( \psi : M_n \to M_m \) is linear, unital, i.e., \( \psi(I_n) = I_m \), and satisfies \( \psi(U(m)) \subseteq U(n) \). Using this observation, one sees that Theorem 1 is equivalent to the following.

**Theorem 2** Let \( \phi : M_n \to M_m \) be a unital linear transformation satisfying \( \phi(U(n)) \subseteq U(m) \). Then \( m \) is a multiple of \( n \) and
\[
\phi(A) = V[(A \otimes I_s) \oplus (A^t \otimes I_r)]V^{-1}
\]
for some fixed \( V \in U(m) \).

By Theorems 1 and 2, one easily deduces the following result of Marcus [5].

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Corollary 3 A linear operator \( \phi \) on \( M_n \) satisfying \( \phi(U(n)) \subseteq U(n) \) must be of the form
\[
A \mapsto VA W \quad \text{or} \quad A \mapsto VA^t W
\]
for some \( V, W \in U(n) \). If, in addition, we assume that \( \phi \) is unital, then \( \phi \) is an (algebra) automorphism or anti-automorphism.

Let \( GL(m) \) be the group of \( m \times m \) invertible matrices. By a result of Auerbach [1] (see [3] for an elementary proof), if \( G \) is a bounded subgroup of \( GL(m) \), then there exists a positive definite matrix \( P \in M_m \) such that \( PGP^{-1} \subseteq U(m) \). So, if \( \phi : M_n \to M_m \) satisfies \( \phi(U(n)) \subseteq G \) for a bounded subgroup \( G \) of \( GL(m) \), then we may apply Theorem 1 to the mapping \( A \mapsto P\phi(A)P^{-1} \) to determine the structure of \( \phi \). Thus, we have the following corollary.

Corollary 4 Suppose \( \phi : M_n \to M_m \) is a linear transformation such that \( \phi(U(n)) \subseteq G \), where \( G \) is a bounded subgroup of \( GL(m) \). Then \( m \) is a multiple of \( n \) and
\[
\phi(A) = LV[(A \otimes I_s) \oplus (A^t \otimes I_r)]L^{-1}
\]
for some fixed \( L \in GL(m) \) and \( V \in U(m) \).

If we just assume that \( \phi(U(n)) \subseteq GL(m) \), the conclusion of Corollary 4 will not hold as shown by the following example.

Example 5 Consider the unital linear \( \phi : M_2 \to M_2 \) defined by
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & ib \\ c & d \end{pmatrix}.
\]
One readily checks that \( \phi(U(2)) \subseteq GL(2) \). However, \( \phi \) does not preserve the rank of matrices, and hence is not of the form (2) with \( L \in GL(2) \) and \( V \in U(2) \).

Marcus and Purves [6, Theorem 2.1] showed that Corollary 3 is valid if we replace \( U(n) \) by \( GL(n) \). One may wonder whether Theorem 1 or Theorem 2 is valid if we replace \( U(m) \) and \( U(n) \) by \( GL(m) \) and \( GL(n) \), respectively. This is not true as shown by the following example, which is a slight modification of [2, Example 4.3 C].

Example 6 Consider the unital linear map \( \phi : M_2 \to M_6 \) defined by
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} aI_3 & bI_3 \\ cI_3 & dI_3 \end{pmatrix} + \left[ 0_3 \oplus \begin{pmatrix} 0 & b & 0 \\ c & 0 & -b \\ 0 & c & 0 \end{pmatrix} \right].
\]
One readily checks that \( \det(\phi(A)) = \det(A)^3 \), and hence \( \phi(GL(2)) \subseteq GL(6) \). However, \( \phi \left( \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \right) \) is not similar to \( -I_3 \oplus I_3 \). Hence, \( \phi \) is not of the form (1) with \( V \in GL(6) \).
2 Proof of Theorem 2

Let $X = [1] \oplus -I_{n-1}$. Since $Y = \phi(X)$ and $\phi(0.6I + 0.8iX) = 0.6I + 0.8iY$ are unitary, it follows that $Y$ is both hermitian and unitary. So we can further assume that $Y = I_k \oplus -I_{m-k}$; otherwise, replace $\phi$ by a mapping of the form $A \mapsto W^*\phi(A)W$ for some $W \in U(m)$ such that $W^*\phi(X)W = Y$. We always assume that

$$\phi(I_n) = I_m \quad \text{and} \quad \phi([1] \oplus -I_{n-1}) = I_k \oplus -I_{m-k} \quad (3)$$

in the rest of the proof. Our result will follow once we establish the following.

**Assertion** There exist $V \in U(m)$ and nonnegative integers $r$ and $s$ with $r + s = k$ such that $V\phi(A)V^*$ is a block matrix $(A_{ij})_{1 \leq i,j \leq n}$, where $A_{ij} = a_{ij}I_s \oplus a_{ji}I_r$ for all $1 \leq i,j \leq n$.

We prove the Assertion by induction on $n \geq 2$. When $n = 2$, consider the matrix $T = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Note that $\phi(T)$, $\phi(0.6I + 0.8iT)$ and $\phi(0.6([1] \oplus [-1]) + 0.8T)$ are all unitary, which is possible if and only if $k = m - k$, i.e. $m = 2k$, and $\phi(T) = \begin{pmatrix} 0 & U \\ U^* & 0 \end{pmatrix}$ for some unitary matrix $U \in U(k)$. We can further assume that $U = I_k$; otherwise, replace $\phi$ by the mapping $A \mapsto (U^* \oplus I)\phi(A)(U \oplus I)$. Next, consider $S = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$. Then $\phi(S)$, $\phi(0.6I + 0.8S)$ and $\phi(0.6([1] \oplus [-1]) + 0.8iS)$ are all unitary, which is possible if and only if $\phi(S) = \begin{pmatrix} 0 & V \\ -V^* & 0 \end{pmatrix}$. Since $\phi(0.6T \pm 0.8iS)$ are also unitary, we see that $V$ is hermitian.

We can further assume that $V = I_s \oplus -I_{k-s}$; otherwise, replace $\phi$ by a mapping of the form $A \mapsto (W^* \oplus W^*)\phi(A)(W \oplus W)$, where $W \in U(m/2)$ satisfies $W^*W = I_s \oplus -I_{k-s}$. As a result, the modified mapping is of the asserted form with $V = I_m$.

Now, suppose the Assertion is true for $n = p \geq 2$, and consider $n = p + 1$. By (3), we have

$$\phi([1] \oplus 0_p) = I_k \oplus 0_{m-k}.$$ 

Moreover, for any $U \in U(p)$ and any $\mu \in C$ with $|\mu| = 1$, we have $\phi([1] \oplus \mu U) \in U(m)$. It follows that $\phi([1] \oplus U) = I_k \oplus \tilde{\phi}(U) \in U(m)$. By induction assumption, there exist $W \in U(m-k)$ and integers $l$ and $s$ such that $m - k = pl$, and for any $A = (a_{ij}) \in M_p$ we have $\tilde{\phi}(A) = W(A_{ij})W^*$, where $A_{ij} = a_{ij}I_s \oplus a_{ji}I_{l-s}$ for all $1 \leq i,j \leq p$. We may assume that $W = I_{m-k}$; otherwise, replace $\phi$ by the mapping $A \mapsto (I_k \oplus W^*)\phi(A)(I_k \oplus W)$. Thus, for any $A = (a_{ij}) \in M_p$, we have

$$\phi([1] \oplus A) = I_k \oplus (A_{ij}), \quad A_{ij} = a_{ij}I_s \oplus a_{ji}I_{l-s}. \quad (4)$$

Now, for $X = 0_p \oplus [1]$, we have

$$\phi(X) = 0_{m-l} \oplus I_l.$$
We can apply the previous argument to $\phi(U \oplus [1])$ for $U \in U(p)$ and conclude that there exist $V \in U(m - l)$ and integers $u, v$ such that $m - l = pu$, and for any $B = (b_{ij}) \in M_p$

$$\phi(B \oplus [1]) = V(B_{ij})V^* \oplus I_l, \quad B_{ij} = b_{ij}I_v \oplus b_{ji}I_{u-v}. \quad (5)$$

Next, consider $X = [1] \oplus 0_{p-1} \oplus [1]$. By (4) and (5), we see that

$$\phi(X) = V[I_u \oplus 0_{m-l-u}]V^* \oplus I_l = I_k \oplus 0_{m-k-l} \oplus I_l.$$ 

Hence $u = k$ and $V = V_1 \oplus U_2$ for some $V_1 \in U(k)$, $U_2 \in U(m - l - k)$. Moreover, from $m - k = pl$ and $m - l = pu$, we have $k = l$ and $m = k(p + 1)$.

Let $E_{ij} \in M_{p-1}$ be the matrix with an 1 at the $(i, j)$-th position and 0 elsewhere. By considering $\phi(X)$ with $X = [1] \oplus E_{ii} \oplus [1]$, we see that $V = V_1 \oplus V_2 \oplus \ldots \oplus V_p$ for some $V_1, \ldots, V_p \in U(k)$. By considering $\phi(X)$ for $X = [1] \oplus E_{ij} + E_{ji} \oplus [1]$, we see that $V = V_1 = \ldots = V_p$. By considering $[1] \oplus E_{ij} \oplus [1]$, we see that $v = s$ and $V_2 = Y_1 \oplus Y_2$ for some $Y_1 \in U(s)$, $Y_2 \in U(k - s)$. We may now assume that $V = I_m$; otherwise, replace $\phi$ by the mapping

$$A \mapsto [V_1 \oplus (I_p \otimes V_2)]^* \phi(A)[V_1 \oplus (I_p \otimes V_2)].$$

Hence, (4) and (5) hold with $V = I_m$; so $\phi(A) = (A_{ij})$ where $A_{ij} = a_{ij}I_s \oplus a_{ji}I_{k-s}$ if $(i, j) \neq (1, p + 1)$ or $(p + 1, 1)$.

Now, apply the previous argument to $\phi(C)$ for those matrices $C \in M_{p+1}$ such that $c_{2j} = c_{i2} = 0$ for $i \neq 2 \neq j$ and $c_{22} = 1$. We see that there exists $X, Y \in U(k)$ so that

$$A_{1,p+1} = X(a_{1,p+1}I_s \oplus a_{p+1,1}I_{k-s})X^* \quad \text{and} \quad A_{p+1,1} = Y(a_{p+1,1}I_s \oplus a_{1,p+1}I_{k-s})X^*.$$ 

The rest of our proof is to show that $X$ and $Y$ may be assumed to be $I_k$. To this end, let

$$U = \begin{pmatrix} 0.6 & 0 \cdots & 0.8 \\ -0.8 & 0 \cdots & 0.6 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \in U(p + 1).$$

Then $\phi(U) \in U(m)$. The submatrix of $\phi(U)$ formed by the first $2k$ rows equals

$$\begin{pmatrix} 0.6I_k & 0 \cdots & 0 & X[0.8I_s \oplus 0_{k-s}]Y^* \\ -0.8I_s \oplus 0_{k-s} & \cdots & \cdots & 0.6I_s \oplus 0_{k-s} \end{pmatrix}$$

and has orthonormal row vectors. Therefore $X[I_s \oplus 0_{k-s}]Y^* = I_s \oplus 0_{k-s}$. Next, considering $U^*$, we have $X[0_s \oplus I_{k-s}]Y^* = 0_s \oplus I_{k-s}$. Thus for $(i, j) = (1, p + 1)$ or $(p + 1, 1)$, we also have $A_{ij} = a_{ij}I_s \oplus a_{ji}I_{k-s}$. The proof of our Assertion is hereby completed, and the theorem follows.

**Note added in proof.**

Professor Peter Šemrl pointed out that Theorem 2 can also be proved by establishing the following.
Lemma 7 If $\phi : M_n \to M_m$ is a unital linear map satisfying $\phi(U(n)) \subseteq U(m)$ then $\phi(H^2) = \phi(H)^2$ for any Hermitian $H \in M_n$.

Proof. Suppose $H \in M_n$ is Hermitian. Then
\[ e^{itH} = I + itH - t^2H^2/2 + \cdots \quad \text{and} \quad \phi(e^{itH}) = I + it\phi(H) - t^2\phi(H^2)/2 + \cdots \]
are unitary. Thus,
\[ I = \phi(e^{itH})\phi(e^{itH})^* = (I + it\phi(H) - t^2\phi(H^2)/2 + \cdots)(I - it\phi(H)^* - t^2\phi(H^2)^*/2 + \cdots). \]
Comparing the coefficients of $t$, we see that $\imath\phi(H) - \imath\phi(H)^* = 0$, i.e., $\phi(H)$ is Hermitian. Now, comparing the coefficient at $t^2$, we see that $-\phi(H^2)/2 + \phi(H)^2 - \phi(H^2)/2 = 0$, i.e., $\phi(H^2) = \phi(H)^2$.

Once this is done, one can follow the proof in [4, Corollary 4.3], which depends on Noether-Skolem Theorem, to conclude that $\phi$ is of the asserted form. In any event, our proof is more straight forward and self-contained.

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References


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