A survey on Linear Preservers of Numerical Ranges and Radii

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Abstract

We survey results on linear operators leaving invariant different kinds of generalized numerical ranges and numerical radii.

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1 Introduction

Let $\mathcal{B}(\mathcal{H})$ be the algebra of bounded linear operators acting on the Hilbert space $\mathcal{H}$. If $\dim \mathcal{H} = n$, we identify $\mathcal{B}(\mathcal{H})$ with $\mathcal{M}_n$, the algebra of $n \times n$ complex matrices. The *numerical range* (or *field of values*) of $A \in \mathcal{B}(\mathcal{H})$ is the set

$$W(A) = \{(Ax, x) : x \in \mathcal{H},\ (x, x) = 1\},$$

and the *numerical radius* of A is the quantity

$$r(A) = \sup\{|z| : z \in W(A)\}.$$

The study of numerical range and numerical radius has a long history (see [8, Chapter 1] and its references). There are many generalizations of these concepts motivated by both theoretical study and applications. A keyword search in MathSciNet will generate hundreds of items. These papers are related to many different subjects in pure and applied sciences. Among the many directions of active research on the numerical range and radius, there has been a great deal of interest in characterizing linear operators satisfying

$$F(T(A)) = F(A) \qquad \text{for all } A \tag{1}$$

where F is a certain kind of generalized numerical range or numerical radius. We say that a linear operator T is a *linear preserver* of F if it satisfies (1).

The first numerical range preserver result was due to Pellegrini [34], who characterized those linear operators T on a unital Banach algebra $\mathcal{A}$ with unity e preserving the numerical range of $A \in \mathcal{A}$ defined by

$$V(A) = \{f(A) : f \in \mathcal{S}\}$$

where

$$\mathcal{S} = \{f \in \mathcal{A}^* : f(e) = 1 = \|f\|\}$$

is the set of *states* of the Banach algebra. In particular, the following result is shown.

(1.1) *A linear operator T on $\mathcal{A}$ satisfies*

$$V(T(A)) = V(A) \qquad \textit{for all } A \tag{2}$$

if and only if the dual transformation T^* satisfies $T^*(\mathcal{S}) = \mathcal{S}$. Furthermore, if $\mathcal{A}$ is a C^*-algebra, then these conditions are equivalent to the fact that T is a C^*-isomorphism; if $\mathcal{A} = \mathcal{B}(\mathcal{H})$ then T is of the form

$$A \mapsto U^*AU \quad \textit{or} \quad A \mapsto U^*A^tU \tag{3}$$

where A^t denotes the transpose of A with respect to a fixed orthonormal basis.

Several remarks are in order in connection with (1.1). First, when $\mathcal{A} = \mathcal{B}(\mathcal{H})$, the set $V(A)$ becomes the closure of $W(A)$. In such case, one can readily show that if T preserves $W(A)$ then T also preserves $V(A)$, and if T is of the form (3) then T preserves $W(A)$. It follows that T preserves $W(A)$ if and only if T preserves $V(A)$, and the preservers are in the form (3) in both cases.

Second, it is interesting to note that numerical range preservers on C^*-algebras are C^*-isomorphisms, which admit the very nice form (3) when $\mathcal{A} = \mathcal{B}(\mathcal{H})$. In contrary, even an isometric isomorphism on a C^*-algebra may not be an C^*-isomorphism!

Third, it is worth noting that the idea of studying linear preservers via the dual transformations is very useful in studying linear preserver problems (see [23, 24]).

As mentioned before, the numerical range and numerical radius have many generalizations. In the study of these generalizations, researchers often considered the corresponding linear preservers as well. In fact, the study of linear preservers often leads to better understanding of the generalized numerical ranges or radii, and the spaces and algebras on which the concepts are defined. The purpose of this paper is to survey the results on numerical ranges and numerical radii preservers.

In our discussion, besides listing the results and problems, we shall comment on the ideas and techniques used by researchers in their study. Many of these ideas have been used to solve other problems; some techniques have been developed into other branch of study. Some earlier linear preserver results on numerical ranges and radii have been surveyed in [35, Chapter 6]. We shall try our best to avoid overlap.

In the next section, we shall mention some general techniques used in the study of linear preserver problems involving numerical ranges and numerical radii. Linear preserver results will be surveyed in the next few sections.

2 General techniques

2.1 Duality technique

As mentioned in the introduction, one interesting idea in [34] is studying the dual transformation of the numerical range or numerical radius preserver. For example, it is easy to prove the following result.

(2.1) *A linear operator T preserves the classical numerical range on $\mathcal{M}_n$ if and only if T^* maps the set $\mathcal{R}$ of matrices of the form xx^* with $x \in \mathbf{C}^n$ satisfying $(x,x) = 1$ onto itself.*

One can then study or use the results concerning those linear operators L mapping the set $\mathcal{R}$ onto itself.

The success of this approach relies on the fact that the dual transformation has a nice structure, which can be determined. Sometimes, one needs to consider both T and T^* simultaneously. In any event, this approach allows one to use additional techniques that come up naturally from the dual problem, and the dual problem itself may be of interest as well.

2.2 Reduction to existing preserver results

Since there are many well studied linear preserver results, one natural approach to linear preserver problems is to show that the linear preservers under investigation will also preserve some other properties or subsets, whose linear preservers have been characterized. One can then apply the existing results to help solve the original problem. For example, if a linear operator T on $\mathcal{M}_n$ satisfies $T(\mathcal{R}) = \mathcal{R}$, where $\mathcal{R}$ is defined as in the last subsection, then T will map the cone of positive semi-definite matrices onto itself. One can then apply the following result of Schneider [38].

(2.2) *A linear operator on $\mathcal{M}_n$ mapping the cone of positive semi-definite matrices onto itself must be of the form $X \mapsto S^*XS$ or S^*X^tS for some invertible S.*

Once we know that a linear preserver of a generalized numerical range or radius is of the form described in (2.2), it is easy to determine the structure of S.

Other common reductions of generalized numerical range or radius preservers lead to the applications of the following results [1, 29] on linear operators mapping the set of rank k (or rank at most k) matrices into itself or mapping the set of unitary matrices into itself.

(2.3) *Let $1 \le k \le n$. A linear operator on $\mathcal{M}_n$ mapping the set of rank k matrices into itself must be of the form $X \mapsto MXN$ or $X \mapsto MX^tN$ for some invertible M and N.*

(2.4) *A linear operator on $\mathcal{M}_n$ mapping the set of unitary matrices into itself must be of the form $X \mapsto UXV$ or $X \mapsto UX^tV$ for some unitary U and V.*

Again, once it is known a generalized numerical range or radius preserver is of the form described in (2.3) or (2.4), one can easily determine its final structure.

2.3 Geometrical techniques

Very often, the reduction of numerical ranges and radii preservers to other types of linear preservers can be done by studying some related geometrical objects. For example, in the study of the classical numerical range and radius preservers on $\mathcal{M}_n$ it is useful to consider the convex sets

$$\{A \in \mathcal{M}_n : W(A) \subseteq (0,\infty)\}, \quad \{A \in \mathcal{M}_n : r(A) \le 1\},$$

and their extreme rays or extreme points. Using the duality techniques, one may also want to consider the convex hull of the sets

$$\mathcal{R} = \{xx^* : x \in \mathbf{C}^n,\ (x,x) = 1\}, \quad \widetilde{\mathcal{R}} = \{\mu X : X \in \mathcal{R},\ \mu \in \mathbf{C},\ |\mu| = 1\}.$$

In many cases, a generalized numerical range or radius preserver (or the dual transformation) will also preserve these sets. If these sets or their extreme points have special structure, then their preservers will be more tractable. In particular, these sets and their generalizations associated with the generalized numerical ranges and radii can be viewed as special subsets of matrices, differentiable manifolds, algebraic sets, or subsets of the projective spaces. Thus, all kinds of geometrical techniques can be used to characterize their preservers. We refer the readers to [24] and [35, Chapter 4] for more details of such ideas. Of course, to use this approach one has to understand the geometrical properties of sets under consideration.

2.4 A group scheme

The following group theory scheme was originated from Dynkin [5]. We describe this approach in the context of numerical range and numerical radius preservers. Let G be the set of linear operators on $\mathcal{M}_n$ that preserve the classical numerical range. Then one can check that G is a group containing the group H of linear operators of the form $X \mapsto U^*XU$ for a given unitary matrix U. Using some algebraic group or Lie group theory, one can (see [7, 37]) determine all the subgroups of $GL(\mathcal{M}_n)$, the group of invertible operators on $\mathcal{M}_n$, that contain H. The list turns out to be quite short. Hence, one can survey the list and decide which group is the group of numerical range or radius preservers.

To use this group scheme in linear preserver problems involving generalized numerical ranges or radii, it is usually relatively easy to show that a certain group on the list is a subgroup of the preserver group. The difficulty arises when one has to show that a larger group cannot be the preserver group. To achieve that, one has to find a specific linear operator T in the larger group, and a particular $A \in \mathcal{M}_n$ so that $F(T(A)) \neq F(A)$ for the given generalized numerical range or radius F. This requires a good understanding of the generalized numerical range and radius on certain matrices.

Let us illustrate the above comment using the preserver group of the numerical radius. It is clear that the H mentioned above is contained in the larger group $\widetilde{H}$ of operators of the form $A \mapsto UAV$ with unitary U and V. To see that $\widetilde{H}$ is not in the preserver group of the classical numerical radius, consider $A = \operatorname{diag}(1, 0, \ldots, 0)$ which has numerical radius one. Let $U = I$, V be the matrix obtained from I by interchanging the first two columns, and define $T \in \widetilde{H}$ by $X \mapsto UXV$. Then $T(A)$ has numerical radius $1/2$, and hence T is not a numerical radius preserver; therefore, $\widetilde{H}$ cannot be in the preserver group. We refer the readers to [7, 26, 27] for more examples on how to use the group theory approach.

3 C-numerical ranges and radii

Let $C \in \mathcal{M}_n$. The *C-numerical range* of $A \in \mathcal{M}_n$ is defined by

$$W_C(A) = \{\operatorname{tr}(CU^*AU) : U^*U = I\},$$

and the *C-numerical radius* of A is the quantity

$$r_C(A) = \max\{|z| : z \in W_C(A)\}.$$

When C is normal with eigenvalues $c_1, \dots, c_n \in \mathbf{C}$, then $W_C(A)$ reduces to the c-numerical range of A defined by

$$W_c(A) = \left\{ \sum_{j=1}^{n} c_j (Ax_j, x_j) : \{x_1, \dots, x_n\} \text{ an orthonormal basis for } \mathbf{C}^n \right\};$$

when C is a rank k orthogonal projection with $1 \le k \le n$, then $W_C(A)$ reduces to the k-numerical range

$$W_k(A) = \left\{ \sum_{j=1}^{k} (Ax_j, x_j) : \{x_1, \dots, x_k\} \text{ an orthonormal set in } \mathbf{C}^n \right\}.$$

Similarly, one can define $r_c(A)$ and $r_k(A)$.

Early study of C-numerical range and radius preservers were surveyed in [35, Chapter 5]. We briefly describe the results and will then move on to the recent development.

3.1 c-numerical range and radius

Note that $W_n(A) = \{\operatorname{tr} A\}$, and the trace preserving maps on $\mathcal{M}_n$ was studied in [11]. Basically, all one can say is the following.

(3.1) *A linear operator T on $\mathcal{M}_n$ preserves the trace function if and only if its dual transformation T^* is unital, i.e., $T^*(I) = I$.*

Using the geometrical properties of $W_k(A)$ and the Fundamental Theorem of Projective Geometry, Pierce and Watkins [36] showed that for $1 \le k < n$ with $k \ne n/2$, linear preservers of $W_k(A)$ are of the standard form (3). The case for $k = n/2$ was left as an open problem.

In [13], this author characterized the set $Ext(\mathcal{S})$ of the extreme points of the convex set

$$\mathcal{S} = \{A \in \mathcal{M}_n : A = A^*,\ r_k(A) \le 1\},$$

and used the fact that a unital k-numerical radius preserver must map the set $Ext(\mathcal{S})$ onto itself to characterize k-numerical radius preservers. As a corollary, the open problem in [36] was answered. We summarize the results in the following.

(3.2) *Suppose $1 \le k < n$ and T is a k-numerical range preserver on $\mathcal{M}_n$. Then there exists a unitary U such that one of the following holds.*

(i) *T is of the standard form*

$$A \mapsto U^*AU \quad \text{or} \quad A \mapsto U^*A^tU.$$

(ii) *$k = n/2$ and T is of the form*

$$A \mapsto (\operatorname{tr} A)I/2 - U^*AU \quad \text{or} \quad A \mapsto (\operatorname{tr} A)I/2 - U^*A^tU.$$

Furthermore, unital k-numerical radius preservers must be multiples of k-numerical range preservers.

In [33], the author studied k-numerical range preservers on $\mathcal{B}(\mathcal{H})$ and solved the problem in [36] using a different approach, namely, he considered the convex cone

$$\mathcal{P} = \{A \in \mathcal{B}(\mathcal{H}) : W_k(A) \subseteq [0, \infty)\}.$$

Li and Tsing [21, 23] established the duality result asserting that a linear operator T on $\mathcal{M}_n$ preserves the c-numerical range if and only if it dual transformation T^* maps the unitary orbit

$$\mathcal{U}(C) = \{U^*CU : U \in \mathcal{M}_n,\ U^*U = I\}$$

of the diagonal matrix $C = \operatorname{diag}(c_1, \ldots, c_n)$ onto itself. Then they used some theory in differential geometry to analyze the set $\mathcal{U}(C)$ and characterized the linear operators mapping $\mathcal{U}(C)$ onto itself. This leads to the following characterization of the c-numerical range and radius preservers (without the unital assumption) for a real vector c.

(3.3) *Let $c = (c_1, \ldots, c_n)$ be a real vector, and T be a c-numerical range preservers. Set $C = \operatorname{diag}(c_1, \ldots, c_n)$. Then there exist a unitary U and $\mu \in \mathbf{R}$ such that the matrices $C_0 = C - (\operatorname{tr} C)I/n$ and μC_0 are unitarily similar, and T is of the form*

$$A \mapsto \mu U^*(A - (\operatorname{tr} A)I/n)U + (\operatorname{tr} A)I/n \quad \textit{or} \quad A \mapsto \mu U^*(A^t - (\operatorname{tr} A)I/n)U + (\operatorname{tr} A)I/n.$$

Furthermore, c-numerical radius preservers are multiples of c-numerical range preservers.

Man [28] further refined the proofs in [21, 23] and extended the results to c-numerical range and radius preservers for complex vectors $c = (c_1, \ldots, c_n)$ with $\sum_{j=1}^n c_j \neq 0$.

Note that c-numerical range preservers always fixes the identity matrix, and its action on the trace zero part $A - (\operatorname{tr} A)I/n$ is always of the form $X \mapsto \mu U^*XU$ or $X \mapsto \mu U^*X^tU$, where μ is a scalar such that C_0 and μC_0 are unitarily similar. It turns out that these are the main features for C-numerical range preservers even for general C as shown in the next subsection.

3.2 C-numerical range and radius

The methods used to deal with the c-numerical range and radius preservers do not seem to be applicable to the C-numerical range and radius preservers for a non-normal matrix C. In [17], the authors made the first attempt to study the general problem by treating the case when C has rank one. Their proofs still depend on the fact that a linear map preserves the C-numerical range if and only if the dual transformation T^* maps the unitary similarity orbit $\mathcal{U}(C)$ onto itself. The following result was obtained.

(3.4) *Suppose $C \in \mathcal{M}_n$ is a rank one matrix.*

(a) *If $\operatorname{tr} C \neq 0$ then C-numerical range preservers must be of the standard form*

$$A \mapsto U^*AU \quad \textit{or} \quad A \mapsto U^*A^tU$$

for some unitary U; C-numerical radius preservers are multiples of C-numerical range preservers.

(b) *If* $\operatorname{tr} C = 0$*, then* C*-numerical range and* C*-numerical radius have the same form, namely,*

$$A \mapsto \mu U^* A U + f(A) I \quad \text{or} \quad A \mapsto \mu U^* A^t U + f(A) I \tag{4}$$

for some unitary U *and linear functional* f *on* $\mathcal{M}_n$.

The proof was computational and used a lot of basic properties of the unitary orbits $\mathcal{U}(C)$. It was shown that if $\operatorname{tr} C \neq 0$ then C-numerical range preservers are also classical numerical range preservers; if $\operatorname{tr} C = 0$ then C numerical range preservers are c-numerical range preservers for $c = \operatorname{diag}(1, -1, 0, \ldots, 0)$. One can then apply known results in [21, 23] to finish the proof.

In [7], the authors determined all possible compact subgroups of $GL(\mathcal{M}_n)$ which contains the group $PSU(n)$ of operators of the form $A \mapsto U^* A U$ for a given unitary U. The result was then used to characterize linear operators which map $\mathcal{U}(C)$ onto itself, and linear preserves of the C-numerical range and C-numerical radius.

(3.5) *Let* $C \in \mathcal{M}$ *be nonzero. Set* $C_0 = C - (\operatorname{tr} C) I/n$*. If* T *is a* C*-numerical range preserver, then there exist* $\mu \in \mathbf{C}$ *and a unitary* U *such that one of the following holds.*

(i) μC_0 *and* C_0 *are unitarily similar, and* T *is of the form* $A \mapsto f(A) I + \mu U^* A U$.

(ii) μC_0^t *and* C_0 *are unitarily similar, and* T *is of the form* $A \mapsto f(A) I + \mu U^* A^t U$,

where f *is a linear functional on* $\mathcal{M}_n$ *satisfying* $f(A) = (1 - \mu)(\operatorname{tr} A)/n$ *in case* $\operatorname{tr} C \neq 0$*. Furthermore,* C*-numerical radius preservers are multiples of* C*-numerical range preservers.*

By the above result, the problem on C-numerical range and radius preservers is completely settled.

3.3 Unitary similarity invariant norms, sets, and functions

A subset $\mathcal{S}$ in $\mathcal{M}_n$ is *unitary similarity invariant* if for any unitary $U \in \mathcal{M}_n$ we have

$$U^* \mathcal{S} U = \{U^* X U : X \in \mathcal{S}\} = \mathcal{S}.$$

It is easy to show that $\mathcal{S}$ is unitary similarity invariant if and only if $\mathcal{S}$ is a union of unitary similarity orbits. A function F on $\mathcal{M}_n$ is *unitary similarity invariant* if for any unitary $U \in \mathcal{M}_n$ we have

$$F(U^* A U) = F(A) \quad \text{for any } A \in \mathcal{M}_n;$$

if F is a norm on $\mathcal{M}_n$ then F is a *unitary similarity invariant norm*. Evidently, every C-numerical radius is a unitary similarity invariant semi-norm. It is known (e.g., see [16]) that $r_C(\cdot)$ is a norm if and only if C is non-scalar and $\operatorname{tr} C \neq 0$. Moreover, for every unitary similarity invariant norm $\|\cdot\|$ on $\mathcal{M}_n$ there exists a compact subset $\mathcal{S} \subseteq \mathcal{M}_n$ such that

$$\|A\| = \max\{r_C(A) : C \in \mathcal{S}\}.$$

In [7], the authors also study the linear preservers of unitary similarity invariant sets and unitary similarity invariant norms. Actually, using the group scheme, one can show that if the collection of the linear preservers of a given unitary similarity invariant set or function forms a compact group G, then there are a limited choices for G. We refer the readers to [7] and [37] for the details.

3.4 Related results and problems

There are several other directions of research on C-numerical range and radius preservers. First, there has been interest in studying linear preservers of the c-numerical range / radius on $\mathcal{B}(\mathcal{H})$ or a general C^*-algebra. A few special cases have been treated (see [2, 3, 33]), but the problem is open in general. For instances, the structures of k-numerical radius preservers, and the c-numerical range preservers for a real vector c are still unknown. Also, it is unclear how to extend the definition of the C-numerical range to an operator $A \in \mathcal{B}(\mathcal{H})$. If $C \in \mathcal{B}(\mathcal{H})$ is a finite rank operator, then one can use the definition

$$W_C(A) = \{\mathrm{tr}\,(CU^*AU) : U^*U = I\}.$$

Also, one may consider an arbitrary linear functional f and define

$$W_f(A) = \{f(U^*AU) : U^*U = I\}.$$

One may also consider the C-numerical range and radius on other types of algebras. For example, Cheung and Li [4] (see also [18]) studied c-numerical range and radius preservers on the algebra $\mathcal{T}(n_1,\dots,n_k)$ of block triangular matrices $A = (A_{ij})_{1\le i,j\le k} \in M_n$ such that $A_{ii} \in \mathcal{M}_{n_i}$ for $i = 1,\dots,k$, and A_{ij} is the zero matrix if $i > j$. They showed that a c-numerical range preserver T on $\mathcal{T}(n_1,\dots,n_k)$ can be extended to a c-numerical range preserver on $\mathcal{M}_n$, and then apply the results on $\mathcal{M}_n$ to determine the structure of T. A similar treatment can be done for c-numerical radius preservers if the sum of the entries of c is nonzero; in such cases, c-numerical radius preservers are multiples of c-numerical range preservers on the algebra of block triangular matrices. We summarize the result in the following.

(3.6) *Let $c = (c_1,\dots,c_n)$ be a real vector. Suppose T is a c-numerical range preserver on $\mathcal{T}(n_1,\dots,n_k)$. Then there exist $\mu = \pm 1$ such that the matrices $C_0 = \mathrm{diag}\,(c_1,\dots,c_n) - (\sum_{j=1}^n c_j)I/n$ and μC_0 are unitarily similar, a linear functional f on $\mathcal{T}(n_1,\dots,n_k)$ such that $f(A) = (\mathrm{tr}\,A)/n$ if $\sum_{j=1}^n c_j \neq 0$, and a unitary $V \in \mathcal{T}(n_1,\dots,n_k)$ such that T is of the form*

$$A \mapsto f(A)I + \mu V^*(A - (\mathrm{tr}\,A)I/n)V \quad \text{or} \quad A \mapsto f(A)I + \mu V^*(A - (\mathrm{tr}\,A)I/n)^+V,$$

where (i) *X^+ denotes X or* (ii) *$n_j = n_{k-j+1}$ for all $1 \le j \le k/2$ and X^+ denotes $X' = EX^tE$, the transpose of X taken with respect to the anti-diagonal $E = E_{1n} + E_{2,n-1} + \cdots + E_{n1}$.*

Furthermore, if $\sum_{j=1}^n c_j \neq 0$, then c-numerical radius preservers on $\mathcal{T}(n_1,\dots,n_k)$ are c-numerical range preservers.

The following problem is still open:

Characterize the c-numerical radius preservers $\mathcal{T}(n_1, \ldots, n_k)$ for a real vector c with entries sum up to zero.

One may also consider C-numerical range and radius preservers on $\mathcal{T}(n_1, \ldots, n_k)$ for a general matrix C, which may be assumed to be in triangular form. Furthermore, one may study similar problems for general nested algebras.

4 Decomposable numerical ranges

Suppose $1 \leq m \leq n$ and $\chi : H \to \mathbf{C}$ is a degree 1 character on a subgroup H of the symmetric group S_m of degree m. The generalized matrix function of $B = (b_{ij}) \in M_m$ associated with χ is defined by

$$d_\chi(B) = \sum_{\sigma \in H} \chi(\sigma) \prod_{j=1}^{m} b_{j,\sigma(j)}.$$

For instance, if χ is the principal character on $H = S_m$, i.e., $\chi(\sigma) = 1$ for all $\sigma \in H$, then $d_\chi(B) = \text{per}(B)$ is the permanent of B; if χ is the alternate character on $H = S_m$ then $d_\chi(B) = \det(B)$ is the determinant of B.

The decomposable numerical range of $A \in \mathcal{M}_n$ associated with χ is defined by

$$W_\chi(A) = \left\{ \frac{d_\chi(X^*AX)}{d_\chi(X^*X)} : X \in M_{n\times m},\ d_\chi(X^*X) \neq 0 \right\},$$

and the decomposable numerical radius is defined by

$$r_\chi(A) = \max\{|\eta| : \eta \in W_\chi(A)\}.$$

In terms of the induced operator of A associated with χ acting on the symmetry classes of tensors defined by χ, we have

$$W_\chi(A) = \{(K(A)x^*, x^*) : x^* \text{ is a decomposable unit tensor }\},$$

see [30, 32] for the background. The decomposable numerical range and radius are useful concepts in the study of induced operators. When $m = 1$, the decomposable numerical range and radius reduce to the classical numerical range and radius.

4.1 Basic results

There has been considerable interest in studying linear preservers of decomposable numerical ranges and radii.

Early study of the decomposable numerical range and radius preservers began with the case when χ is the alternate character on S_m. In such a case, the decomposable numerical range reduces to the mth determinantal range

$$W_m^\wedge(A) = \{\det(X^*AX) : X \text{ is } n \times m,\ \det(X^*X) = 1\}.$$

When $n = m$, $W_n^\wedge(A) = \{\det(A)\}$, and Frobenius [6] proved that

(4.1) *A linear preserver of the determinant function on $\mathcal{M}_n$ must be of the form*

$$A \mapsto MAN \quad \text{or} \quad A \mapsto MA^tN \tag{5}$$

for some $M, N \in M_n$ with $\det(MN) = 1$.

For $1 \leq m < n$, Marcus and Filippenko [31] showed that

(4.2) *A linear preserver of $W_m^\wedge(A)$ on $\mathcal{M}_n$ must be of the form*

$$A \mapsto \xi U^*AU \quad \text{or} \quad A \mapsto \xi U^*A^tU \tag{6}$$

for some unitary U and $\xi \in \mathbf{C}$ with $\xi^m = 1$.

Their proof was done by showing that a determinantal range preserver must map the set of unitary matrices into itself. It will then follow from a result of Marcus [29] that T is of the form

$$A \mapsto UAV \quad \text{or} \quad A \mapsto UA^tV$$

for some unitary U and V. The proof was then finished by showing that $UV = \xi I$ for some $\xi \in \mathbf{C}$ with $\xi^m = 1$. In [22], the authors characterized $r_m^\wedge(\cdot)$ preservers.

(4.3) *Let $1 \leq m \leq n$. Linear preservers of $r_m^\wedge(\cdot)$ are multiples of $W_m^\wedge(\cdot)$ preservers.*

The proof was done by showing that $r_m^\wedge(\cdot)$ preservers are multiples of determinant preservers. Thus, they must be of the form (5) with $|\det(MN)| = 1$. It was then shown that M and N must be unitary satisfying $MN = \xi I$ with $\xi^m = 1$ if $1 \leq m < n$. Tam [41] has also studied determinantal radius preservers with the additional unital assumption, i.e., the linear map fixes the identity matrix.

In [40], Tam considered the case when $H = S_m$ and χ is the principal character, In such a case, he showed that a decomposable numerical range preserver must be a multiple of a linear operator mapping the set of positive semi-definite matrices onto itself. It then follows [38] that T is of the form

$$A \mapsto \xi S^*AS \quad \text{or} \quad A \mapsto \xi S^*A^tS$$

for some invertible S and $\xi \in \mathbf{C}$. Finally, it was shown that $\xi^m = 1$ and S is unitary. In [42], Tam attempted to extend the results in [31] and [40] to some more general situations. Unfortunately, there was a gap in his proof as pointed out in [27, Remark 2.14].

The complete solution of the problem on decomposable numerical range and radius preservers was recently done in [27]. First, the authors identified some characters χ such that the induced matrix $K(A)$ is of the form $\det(A)^k I_N$ for some positive integers k and N. Such a character is referred to as of the *determinant type*. If χ is of the determinant type then

$$W_\chi(A) = \{\det(A)^k\} \quad \text{and} \quad r_\chi(A) = |\det(A)|^k,$$

and we have the following result.

(4.4) *Suppose χ is of the determinant type so that $K(A) = \det(A)^k I_n$ for all $A \in \mathcal{M}_n$. Then linear preservers of $W_\chi(A)$ must be of the form (5) for some $M, N \in \mathcal{M}_n$ satisfying* $\det(MN)^k = 1$. *Furthermore, decomposable numerical radius preservers must be multiples of decomposable numerical range preservers.*

For all other cases, we have the following [27].

(4.5) *Suppose χ is not of the determinant type. Then decomposable numerical range preservers must be of the form (6) for some unitary $U \in \mathcal{M}_n$ and $\xi \in \mathbf{C}$ with $\xi^m = 1$. Furthermore, decomposable numerical radius preservers are multiples of decomposable numerical range preservers.*

The key step of the proof of decomposable numerical range preservers is to show that they always map the set of positive semi-definite matrices onto itself, and therefore the result of Schneider [38] is applicable. The proof of decomposable numerical radius was done by the group scheme.

4.2 Decomposable numerical range and radius on orthonormal tensors

In some study, one would like to restrict the choice of the unit decomposable tensors in the definition of the decomposable numerical range and radius to unit tensors arising from orthonormal vectors. More precisely, for $1 \le m \le n$ and a degree one character χ on a subgroup of S_m, one defines the decomposable numerical range of $A \in \mathcal{M}_n$ on orthonormal tensors associated with χ by

$$W_\chi^\perp(A) = \{d_\chi(X^*AX) : X \text{ is an } n \times m \text{ matrix such that } X^*X = I_m\}$$

and the decomposable numerical radius of A by

$$r_\chi^\perp(A) = \max\{|z| : z \in W_\chi^\perp(A)\}.$$

It is interesting to note that if χ is the alternate character on $H = S_m$, then $W_\chi(A) = W_\chi^\perp(A)$. Again, we consider the linear preserver results.

Hu and Tam [9, 10] (see [7] for the correction of the statement in [9, Theorem 6]) proved the following result when χ is the principal character on $H < S_m$.

(4.6) *A linear operator T on M_n is a preserver of $W_\chi^\perp$ if and only if there exist a unitary matrix $U \in M_n$ and $\xi \in \mathbf{C}$ with $\xi^m = 1$ such that*
(i) *T is of the form described in (6), or*
(ii) *$m = n = 2$, $H = S_2$ and T is of the form*

$$A \mapsto \xi[U^*AU + (\pm i - 1)(\operatorname{tr} A)I/2] \quad \text{or} \quad A \mapsto \xi[U^*A^tU + (\pm i - 1)(\operatorname{tr} A)I/2].$$

Very recently, Li and Zaharia [25] completed the study by proving the following theorem.

(4.7) *Suppose χ is not the principal character and $W(A) \neq \{\det(A)\}$. A linear preserver of $W_\chi^\perp$ on $\mathcal{M}_n$ must be of the form described in (6) for some unitary U and $\xi \in \mathbf{C}$ with $\xi^m = 1$.*

For both (4.6) and (4.7), the proofs were done by studying the condition under which $W_\chi^\perp(A)$ is a subset of real numbers, and then using the results to show that a $W_\chi^\perp$ preserver must map the set of positive semi-definite matrices onto itself, hence the result of Schneider [38] applicable.

Next we turn to the results on the linear preservers of $r_\chi^\perp$. For a long time, the only known results are for the cases when $m = 1$ [14] and when χ is the alternate character on $H = S_m$ [22] (see also [41]). In these cases, the linear preservers of $r_\chi^\perp$ are always multiples of $W_\chi^\perp$ preservers. Similar to the other cases, even if the linear preservers of a generalized numerical range are determined, it always requires different (and usually more difficult) techniques to characterize the linear preservers of the generalized numerical radius (see e.g. [35, Chapter 5]).

In [26], the authors used the group scheme to confirm that:

(4.8) *Let $1 \leq m \leq n$ and χ be a degree one character on $H < S_m$. Except for the case: $m = n > 3$ and $r_\chi^\perp(A) \neq |\det(A)|$, linear preservers of $r_\chi^\perp$ on $\mathcal{M}_n$ are multiples of linear preservers of $W_\chi^\perp$.*

It was conjectured that the same conclusion should hold for the exceptional case, and some partial results were presented in [26].

4.3 Related results and problems

There are other types of operators acting on the symmetry classes of tensors. For example, consider the rth derivation $D_r(A)$ of $A \in \mathcal{M}_n$ acting on the symmetry class of tensors associated with χ, where $D_k(A)$ is given by the formula

$$K(I + tA) = \sum_{k=0}^{m} t^r D_k(A).$$

One may consider the decomposable numerical range and radius of $D_r(A)$, and study the corresponding linear preserver problems. The special case for $K(A) = C_m(A)$, the mth compound matrix of A, and $1 \leq k \leq m$, the decomposable numerical range reduces to (k, m) numerical range defined by

$$W_{k,m}(A) = \{\operatorname{tr} C_k(X^*AX) : X \text{ is } n \times m,\ X^*X = I_m\}.$$

When $m = 1$, this reduces to the k-numerical range; when $m = k$, this reduces to the kth determinantal range. Linear preservers of $W_{k,m}(A)$ and $r_{k,m}(A)$ were characterized in [20] and [15], respectively.

(4.9) *Let $1 \le k < m \le n$. Linear preservers of the (k,m) numerical range on $\mathcal{M}_n$ must be of the form (6) for some unitary U and $\xi \in \mathbf{C}$ with $\xi^m = 1$. Moreover, linear preservers of the (k,m) numerical radius are multiples of linear preservers of the (k,m) numerical range.*

The proof of the (k,m) numerical range preservers was done by showing that a multiple of the preserver will map the set of rank one Hermitian matrices to itself, and hence it is of the form

$$A \mapsto \xi S^* A S \quad \text{or} \quad A \mapsto \xi S^* A^t S$$

for some invertible matrix S and $\xi \in \mathbf{C}$. One then easily deduces the properties on S and ξ. The proof of the (k,m) numerical radius preservers was done by skillful computation and the knowledge of the (k,m) numerical radius for some special matrices A.

More generally, the derivation $D_\tau(A_1, \dots, A_p)$ of $A_1, \dots, A_k \in \mathcal{M}_n$ is defined by

$$K\left(\sum_{j=1}^p t_j A_j\right) = \sum_\tau t_1^{r_1} \cdots t_p^{r_p} D_\tau(A_1, \dots, A_p),$$

where the summation on the right is over all partitions $\tau : r_1 + \cdots + r_p = m$, see [30, Chapter 3]. Again, one may define the decomposable numerical range and radius of $D_\tau(A_1, \dots, A_p)$, and study the corresponding linear preserver problems. Not much has been done in this general context.

It is possible to consider tensor powers of $\mathcal{B}(\mathcal{H})$ or exterior powers of $\mathcal{B}(\mathcal{H})$. One may consider the decomposable numerical range of operators in $\mathcal{B}(\mathcal{H})$, and the corresponding linear preserver problems.

5 Additional results and problems

5.1 Unitary congruence and numerical range

A variation of the C-numerical range is the *C-congruence numerical range* defined by

$$\widetilde{W}_C(A) = \{\mathrm{tr}\,(CU^tAU) : U \text{ unitary }\}.$$

One can define the corresponding *C-congruence numerical radius* $\tilde{r}_C(A)$. These concepts can also be considered in the spaces of $n \times n$ symmetric matrices or skew-symmetric matrices. We refer the readers to [12] for some basic definitions and references. Similar to the treatment in Section 3, one can consider the *unitary congruence orbit* of $C \in \mathcal{M}_n$ defined by

$$\widetilde{\mathcal{U}}(C) = \{U^tCU : U \in \mathcal{M}_n,\ U^*U = I\},$$

unitary congruence invariant sets $\mathcal{S}$ that satisfy

$$U^t\mathcal{S}U = \{U^tXU : X \in \mathcal{S}\} = \mathcal{S} \quad \text{whenever } U \text{ is unitary},$$

unitary congruence invariant functions F that satisfy

$$F(U^tAU) = F(A) \quad \text{whenever } U \text{ is unitary and } A \in \mathcal{M}_n,$$

and *unitary congruence invariant norms* on $\mathcal{M}_n$.

If one regards $A \in \mathcal{M}_n$ as a linear operator on $\mathbf{C}^n$, then clearly U^*AU represents the same linear operator with respect to a different orthonormal basis. Similarly, if one regards $A \in \mathcal{M}_n$ as a bilinear form on $\mathbf{C}^n \times \mathbf{C}^n$ such that $(x, y) \mapsto y^tAx$, then U^tAU represents the same bilinear form with respect to a different orthonormal basis. Hence, it is of interest to study unitary congruence invariant sets, functions and norms. Again, every unitary congruence invariant set is a union of unitary congruence orbits, and every unitary congruence invariant (semi-)norm $\|\cdot\|$ admits a representation in terms of the C-congruence numerical radii, namely, for every unitary congruence invariant norm $\|\cdot\|$ on $\mathcal{M}_n$ there exists a compact subset $\mathcal{S} \subseteq \mathcal{M}_n$ such that

$$\|A\| = \max\{\tilde{r}_C(A) : C \in \mathcal{S}\}.$$

This gives good motivations to study unitary congruence orbit, C-congruence numerical ranges which are images of the unitary congruence orbit of C under a linear functional, and C-congruence numerical radii.

Concerning linear preservers of these concepts, the group theory scheme turns out to be very effective. In [7], the authors determine all the possible subgroups of operators on $\mathcal{M}_n$ that contain the group $\widetilde{H}$ of operators of the form $A \mapsto U^tAU$ where U is unitary. The results can be applied to solve linear preserver problems related to unitary congruence invariant concepts efficiently. Instead of giving a long list of linear preserver theorems, we state a general result [7, Corollaries 3.2 and 3.3] using the following notations.

$SU(n)$: the group of special unitary operators on $\mathbf{C}^n$, i.e., the set of unitary matrices A satisfying $\det(A) = 1$.

$SU(\mathcal{M}_n)$: the group of special unitary operators on $\mathcal{M}_n$.

$SU(\mathcal{S}_n)$: the group of special unitary operators on $n \times n$ symmetric matrices.

$SU(\mathcal{K}_n)$: the group of special unitary operators on $n \times n$ skew-symmetric matrices.

H: the group of operators on $\mathcal{M}_n$ of the form $A \mapsto U^tAU$ where U is unitary.

H_1: the group of operators on $n \times n$ symmetric matrices of the form $A \mapsto U^tAU$ where U is unitary.

H_2: the group of operators on $n \times n$ skew-symmetric matrices of the form $A \mapsto U^tAU$ where U is unitary.

$O(6, \mathbf{R})$: the group of real orthogonal operators acting on the six dimensional real linear space of 4×4 complex skew-symmetric matrices.

$SO(6, \mathbf{R})$: the group of operators in $O(6, \mathbf{R})$ with determinant one.

(5.1) *Suppose G is a compact group of invertible operators on $\mathcal{M}_n$ containing H. Then $G = YZ$ where Y satisfies one of the following conditions* (a) – (d)*, and Z is a subgroup of the centralizer of Y in $\mathcal{U}(\mathcal{M}_n)$.*

(a) *Y is a D-conjugation of the group $SU(\mathcal{M}_n)$, i.e., $L^{-1}SU(\mathcal{M}_n)L$ for some $L \in D$.*

(b) *Y is a D-conjugation of the group generated by operators of the form $A \mapsto UAV$ with $U, V \in SU(n)$, or possibly with the the transposition operator $A \mapsto A^t$.*

(c) *Y is one of the following: $H, H_1 \times H_2, H_1 \times SU(\mathcal{K}_n), SU(\mathcal{S}_n) \times H_2, SU(\mathcal{K}_n) \times SU(\mathcal{S}_n)$.*

(d) *$n = 4$ and $Y = Y_1 \otimes Y_2$, where $Y_1 = H_1$ or $SU(\mathcal{S}_4)$, and $Y_2 = SO(6, \mathbf{R})$ or $Y_2 = O(6, \mathbf{R})$.*

Using the above result, one can easily solve linear preserver problems involving unitary congruence invariant concepts.

5.2 Decomposable C-(decomposable) numerical ranges and radii

In [39, 12], the authors considered the C-decomposable numerical range and the C-congruence decomposable numerical range of $A \in \mathcal{M}_n$ associated with a degree one character χ (see Section 4) defined by

$$W_C^\chi(A) = \{\mathrm{tr}\, K(CU^*AU) : U \text{ unitary } \}$$

and

$$\widetilde{W}_C^\chi(A) = \{\mathrm{tr}\, K(CU^tAU) : U \text{ unitary } \}.$$

One can also consider the corresponding generalized numerical radii. So far, the known results are direct generalizations of those on C-numerical ranges and determinantal ranges. Good motivations for the study of these concepts have yet to be found. If there are reasons for studying these generalizations, one may consider the corresponding linear preserver problems.

5.3 Normed numerical ranges

It is somewhat interesting that the study of numerical range preservers rooted from the paper of Pellegrini [34], who considered numerical range in a Banach space. However, not much follow up work was done. Very recently, Li and Sourour [19] investigated the normed numerical range of $A \in \mathcal{M}_n$ associated with a symmetric norm (or symmetric gauge function) ν on $\mathbf{C}^n$ defined by

$$W(A) = \{(Ax, y) : \nu(x) = \nu^D(y) = 1 = (x, y)\},$$

where $\nu^D(y) = \max\{|(x, y)| : x \in \mathbf{C}^n,\ \nu(x) \le 1\}$ is the dual norm of ν. If we regard $\mathcal{M}_n$ as a unital Banach algebra equipped with the operator norm induced by ν, then convex hull of $W(A)$ reduces to $V(A)$ considered by Pellegrini. In [19], the authors characterized $\nu-$Hermitian matrices in $\mathcal{M}_n$, i.e., those matrices A such that $W(A) \subseteq \mathbf{R}$. The result was used to show that if ν is not a multiple of the $\ell_1, \ell_2, \ell_\infty$ norm, then linear preservers of $W(A)$ or $V(A)$ must be of the form $A \mapsto P^*AP$ where P is the product of a diagonal unitary matrix and a permutation matrix; and normed numerical radius preservers are multiples of normed numerical range preservers. The case when ν is a multiple of ℓ_2 norm reduces to the classical numerical range and radius preserver problems. For the multiples of ℓ_1 and ℓ_∞ norm cases, the structure of numerical range preservers are different, and the corresponding radius preservers are not multiples of range preservers. One may also consider normed numerical range and radius arising from other norms.

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