Schur product of matrices and numerical radius (range) preserving maps

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Dedicated to Professor Roger Horn on the occasion of his sixty fifth birthday.

Abstract

Let $F(A)$ be the numerical range or the numerical radius of a square matrix A. Denote by $A \circ B$ the Schur product of two matrices A and B. Characterizations are given for mappings on square matrices satisfying $F(A \circ B) = F(\phi(A) \circ \phi(B))$ for all matrices A and B. Analogous results are obtained for mappings on Hermitian matrices.

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1 Introduction

Let M_n be the algebra of $n \times n$ complex matrices. Denote the numerical range and numerical radius of $A \in M_n$ by

$$W(A) = \{x^*Ax : x \in \mathbb{C}^n,\ x^*x = 1\} \qquad \text{and} \qquad r(A) = \max\{|\mu| : \mu \in W(A)\}.$$

There has been considerable interest in studying the structure of maps preserving the numerical range or radius. Suppose $U \in M_n$ is a unitary matrix. Define the map ϕ on M_n by

$$A \mapsto U^*AU \quad \text{or} \quad A \mapsto U^*A^tU. \tag{1.1}$$

Then ϕ is a C^*-isomorphism on the C^*-algebra M_n, and a Jordan isomorphism on the Jordan algebra H_n of Hermitian $n \times n$ matrices. Evidently, ϕ is bijective linear and preserves the numerical range, i.e., $W(\phi(A)) = W(A)$ for all A. Pellegrini [14] (see also [11]) obtained an interesting result on numerical range preserving maps on a general C^*-algebra, which implies that a linear map $\phi : M_n \to M_n$ preserving the numerical range must be of this form. One easily deduces that the conclusion is also valid for linear maps ϕ defined on H_n. In [4], it was shown that a multiplicative map $\phi : M_n \to M_n$ satisfies $W(\phi(A)) = W(A)$ for all A if and only if ϕ has the form $A \mapsto U^*AU$ for some $U \in M_n$. In [7], the authors replaced the condition that "ϕ is multiplicative and preserves the numerical range" on the surjective map $\phi : M_n \to M_n$ by the condition that "$W(AB) = W(\phi(A)\phi(B))$ for all A, B", and showed that such a map has the form $A \mapsto \pm U^*AU$ for some unitary operator $U \in M_n$. They also showed that a surjective map $\phi : M_n \to M_n$ satisfies $W(ABA) = W(\phi(A)\phi(B)\phi(A))$ for all $A, B \in M_n$ if and only if ϕ has the form $A \mapsto \mu U^*AU$ or $A \mapsto \mu U^*A^tU$ for some unitary operator $U \in M_n$ and $\mu \in \mathbb{C}$ with $\mu^3 = 1$. Similar results for

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mappings on H_n were also obtained. It is interesting to note that all the results mentioned above show that under rather mild assumptions, a numerical range preserving map ϕ on $\mathbf{V} = M_n$ or H_n must be a multiple of the standard map (1.1).

There is also interest in studying numerical radius preserving maps on matrices or operators. In [9] (see also [2]), it is shown that linear preservers of the numerical radius on $\mathbf{V} = M_n$ or H_n have the form

$$A \mapsto \mu U^* A U \quad \text{or} \quad A \mapsto \mu U^* A^t U$$

for some unitary U and scalar μ with $|\mu| = 1$. By the result in [4], if $\phi : \mathbf{V} \to \mathbf{V}$ is a multiplicative preserver of the numerical radius, then ϕ has the form

$$A \mapsto \mu U^* A U \quad \text{or} \quad A \mapsto \mu U^* \overline{A} U$$

for some unitary $U \in M_n$ and unit scalar μ. By the result in [1], if $\phi : \mathbf{V} \to \mathbf{V}$ satisfies

$$r(\phi(A) - \phi(B)) = r(A - B) \quad \text{for all } A, B \in \mathbf{V},$$

then ϕ has the form

$$A \mapsto \mu U^* A^\tau U + R$$

for some unit scalar μ, $R \in \mathbf{V}$, and unitary $U \in M_n$, where A^τ denotes A, A^t, $\overline{A}$, or A^*.

In this paper, we consider the Schur product (also known as the Hadamard product) of matrices defined by $(a_{ij}) \circ (b_{ij}) = (a_{ij} b_{ij})$, which is quite different from the other types of binary products on $\mathbf{V}$. One easily sees that mappings ϕ in the form (1.1) will not always satisfy

$$W(A \circ B) = W(\phi(A) \circ \phi(B)) \qquad \text{for all } A, B \in \mathbf{V} \tag{1.2}$$

unless the matrix in (1.1) is carefully chosen, say, U is a permutation matrix. On the other hand, if a permutation matrix P is given, and a diagonal unitary matrix D_A is assigned to each $A \in \mathbf{V}$, then a mapping ϕ of the form

$$A \mapsto D_A^* P^t A P D_A \quad \text{or} \quad A \mapsto D_A^* P^t A^t P D_A$$

will satisfy (1.2). A more obscure operation is to choose a matrix $R \in \mathbf{V}$ so that $R \circ R = (\bar{x}_i x_j)$ with $|x_1| = \cdots = |x_n| = 1$ and define the map ϕ by $A \mapsto R \circ A$. Then

$$\phi(A) \circ \phi(B) = R \circ R \circ (A \circ B) = D_x^* (A \circ B) D_x,$$

where D_x is the diagonal matrix with diagonal entries $x_1, \ldots, x_n$, and hence ϕ satisfies (1.2). It turns out that the composition of the maps described above will be the totality of maps satisfying (1.2); see Theorem 1.2.

Of course, a mapping ϕ satisfying (1.2) will also satisfy

$$r(A \circ B) = r(\phi(A) \circ \phi(B)) \qquad \text{for all } A, B \in \mathbf{V}. \tag{1.3}$$

But there may be more admissible maps. For example, the mappings $A \mapsto \overline{A}$ and $A \mapsto A^*$ also satisfy (1.3). Also, if a unit scalar μ_A is assigned to each $A \in \mathbf{V}$, then the mapping $A \mapsto \mu_A A$ also satisfies (1.3). More generally, whenever A is permutationally similar to a direct sum of square matrices of smaller sizes, say, $A_1 \oplus \cdots \oplus A_k$, one can take a pair of diagonal unitary matrices D_A, E_A so that $D_A E_A \in \mathbf{V}$ and $D_A E_A A = A D_A E_A$ (equivalently, $D_A A E_A$ is permutationally similar to $\mu_1 A_1 \oplus \cdots \oplus \mu_k A_k$ for some unit scalars $\mu_1, \ldots, \mu_k$) and define $\phi(A) = D_A A E_A$. Since $A \circ B$ will be permutationally similar to a matrix of the form

$$(A_1 \circ B_{11}) \oplus \cdots \oplus (A_k \circ B_{kk}),$$

and

$$r(X_1 \oplus \cdots \oplus X_k) = \max\{r(X_1), \ldots, r(X_k)\},$$

we see that mappings constructed as above also satisfy (1.3). We will show that these are the only additional maps needed to generate (by compositions) all of the maps satisfying (1.3). Specifically, we have the following theorems (where $n \geq 2$ to avoid trivialities).

Theorem 1.1. *Let $\mathbf{V} = M_n$ or H_n, and let $\phi : \mathbf{V} \to \mathbf{V}$. Then $r(A \circ B) = r(\phi(A) \circ \phi(B))$ for all $A, B \in \mathbf{V}$ if and only if there is a fixed permutation matrix P, a matrix $R \in \mathbf{V}$ such that $R \circ R = (\bar{x}_i x_j)$ with $|x_1| = \cdots = |x_n| = 1$, and a mapping $A \mapsto (D_A, E_A)$ assigning each $A \in \mathbf{V}$ to a pair of diagonal unitary matrices D_A, E_A satisfying $D_A E_A \in \mathbf{V}$ and $D_A E_A A = A D_A E_A$ such that ϕ has the form*

$$X \mapsto R \circ (P^t D_X X^\tau E_X P) \qquad \text{for all } X \in \mathbf{V},$$

where X^τ denotes $X, \overline{X}, X^t$, or X^*. (Of course, $X = X^*$ and $\overline{X} = X^t$ if $\mathbf{V} = H_n$.)

We note again that the condition on D_A and E_A simply means that if Q is a permutation matrix such that $Q^t A Q = A_1 \oplus \cdots \oplus A_m$, then D_A and E_A are chosen such that $Q^t D_A E_A Q = \lambda_1 I \oplus \cdots \oplus \lambda_m I$ accordingly.

Theorem 1.2. *Let $\mathbf{V} = M_n$ or H_n, and let $\phi : M_n \to M_n$. Then $W(A \circ B) = W(\phi(A) \circ \phi(B))$ for all $A, B \in \mathbf{V}$ if and only if there is a fixed permutation matrix P, a matrix $R \in \mathbf{V}$ such that $R \circ R = (\bar{x}_i x_j)$ with $|x_1| = \cdots = |x_n| = 1$, and a mapping $A \mapsto D_A$ from $\mathbf{V}$ to the group of diagonal unitary matrices such that ϕ has the form*

$$X \mapsto R \circ (P^t D_X^* X D_X P) \quad \text{or} \quad X \mapsto R \circ (P^t D_X^* X^t D_X P).$$

The sufficiencies of the theorems are clear by our discussion before the statements. We will prove the necessities in the next two sections.

In our discussion, $|v|$ denotes the vector obtained from $v \in \mathbb{C}^n$ by replacing each entry by its absolute value; $|A|$ has a similar meaning for $A \in M_n$. A vector or a matrix is said to be unimodular if all entries have moduli one. The matrix in M_n whose every entry is one is denoted by J. We say that a vector or a matrix has support at certain entries if all other entries of the vector or matrix equal zero. A matrix A is decomposable if it is permutationally similar to a direct sum of square matrices of smaller sizes; otherwise, A is indecomposable. The Schur-inverse of A is denoted by $A^{(-1)}$, and is defined by $(A^{(-1)})_{ij} = A_{ij}^{-1}$ if $A_{ij} \neq 0$, and $(A^{(-1)})_{ij} = 0$ if $A_{ij} = 0$. Denote by $\{E_{11}, E_{12}, \ldots, E_{nn}\}$ the standard basis for M_n.

2 Proofs for complex matrices

2.1 Auxiliary results

Lemma 2.1. *Suppose $\mathcal{S} \subseteq M_n$ has n^2 nonzero elements such that $X \circ Y = 0$ for any $X \neq Y \in \mathcal{S}$. Then there are nonzero scalars $\mu_{ij} \in \mathbb{C}$ such that*

$$\mathcal{S} = \{\mu_{ij} E_{ij} : 1 \leq i, j \leq n\}.$$

Lemma 2.2. *Let A be a nonnegative matrix such that $A + A^t$ is irreducible. Let U be a unimodular matrix (i.e., $|U_{ij}| = 1$ for all i, j). If $r(A) = r(A \circ U)$ then there exist some unit scalar μ and unimodular vector w such that $A \circ U = A \circ (\mu w w^*)$.*

Proof. Let $x \in \mathbb{R}^n$ be the unique positive unit eigenvector of $(A + A^t)/2$, so $r(A) = x^t A x$. Write $\tilde{A} = A \circ U$. Let $v \in \mathbb{C}^n$ be a unit vector such that $r(\tilde{A}) = |v^* \tilde{A} v|$. Let D be a diagonal unitary such that $D|v| = v$. Then

$$r(\tilde{A}) = ||v|^t D^* \tilde{A} D |v|| \leq |v|^t |D^* \tilde{A} D| |v| \leq x^t A x = r(A),$$

so all the inequalities are in fact equalities. For the second inequality to be equality implies that $|v| = x$ has strictly positive entries. For the first inequality to be equality implies that there exists $\mu \in \mathbb{C}$ with $|\mu| = 1$ such that $D^* \tilde{A} D = \mu A$. If $D = \operatorname{diag}(w)$, then $\tilde{A} = A \circ (\mu w w^*)$ as desired. ■

Lemma 2.3. *Let w, z be complex numbers of modulus one. Then*

$$r\left(\begin{bmatrix} 1 & 1 \\ 0 & w \end{bmatrix}\right) = r\left(\begin{bmatrix} 1 & 1 \\ 0 & z \end{bmatrix}\right) \iff w = z \text{ or } w = \overline{z}.$$

Proof. Let $v = \begin{bmatrix} x & y \end{bmatrix}^t$ and $f(\theta) = r\left(\begin{bmatrix} 1 & 1 \\ 0 & e^{i\theta} \end{bmatrix}\right) = r\left(\begin{bmatrix} e^{-i\theta/2} & 1 \\ 0 & e^{i\theta/2} \end{bmatrix}\right)$. Then

$$\begin{aligned} f(\theta) &= \max\left\{\left| v^* \begin{bmatrix} e^{-i\theta/2} & 1 \\ 0 & e^{i\theta/2} \end{bmatrix} v \right| : \|v\| = 1\right\} \\ &= \max\{|\cos(\theta/2) + \overline{x} y + i(|y|^2 - |x|^2)\sin(\theta/2)| : |x|^2 + |y|^2 = 1\} \\ &= \max\{|\cos(\theta/2) + i(y^2 - x^2)\sin(\theta/2)| + xy : x^2 + y^2 = 1; x, y \geq 0\}. \end{aligned}$$

Let $y = \cos(\alpha/2)$, $x = \sin(\alpha/2)$, $0 \leq \alpha \leq \pi$. We get

$$f(\theta) = \max\{|\cos(\theta/2) + i(\cos\alpha)\sin(\theta/2)| + (1/2)\sin\alpha : \alpha \in [0, \pi]\}.$$

Let $t = \sin\alpha$. We get

$$f(\theta) = \max\left\{\sqrt{1 - \sin^2(\theta/2)t^2} + t/2 : 0 \leq t \leq 1\right\}.$$

Since $\begin{bmatrix} 1 & 1 \\ 0 & e^{i\theta} \end{bmatrix}$ is not normal, its numerical range contains the eigenvalue 1 in its interior, so $f(\theta) > 1$. Thus the maximum is attained at some $t > 0$, and hence f is strictly increasing on $[-\pi, 0]$. Since $f(-\theta) = f(\theta)$, the result follows. ■

Lemma 2.4. *Let w, z be complex numbers of modulus one. Then*

$$r\left(\begin{bmatrix}1 & 1\\ w & 0\end{bmatrix}\right) = r\left(\begin{bmatrix}1 & 1\\ z & 0\end{bmatrix}\right) \iff w = z \text{ or } w = \overline{z}.$$

Proof. Let $f(\theta) = r\left(\begin{bmatrix}1 & 1\\ e^{i\theta} & 0\end{bmatrix}\right) = r\left(\begin{bmatrix}1 & e^{i\theta/2}\\ e^{i\theta/2} & 0\end{bmatrix}\right)$ and $v = \begin{bmatrix}x & y\end{bmatrix}^t$. Then for $\theta \in (-\pi, 0)$,

$$\begin{aligned} f(\theta) &= \max\{|v^* \begin{bmatrix}1 & e^{i\theta/2}\\ e^{i\theta/2} & 0\end{bmatrix} v| : \|v\| = 1\} \\ &= \max\{||x|^2 + 2\mathrm{Re}(\overline{x}y)e^{i\theta/2}| : \|v\| = 1\} \\ &= \max\{|x^2 + 2xye^{i\theta/2}| : x^2 + y^2 = 1; x, y \geq 0\}. \end{aligned}$$

Since $\begin{bmatrix}1 & 1\\ e^{i\theta} & 0\end{bmatrix}$ has unimodular determinant, one eigenvalue has modulus at least one. As this matrix is not normal (except when $e^{i\theta}$ is real, which we've excluded), this eigenvalue lies in the interior of the numerical range, so $f(\theta) > 1$. Thus the maximum does not occur when $x = 0$ or $y = 0$, and hence f is strictly increasing on $[-\pi, 0]$. Since $f(-\theta) = f(\theta)$, the result follows. ■

Lemma 2.5. *Let w, z be complex numbers of modulus one. Then*

$$r\left(\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & w\\ 0 & 0 & 0\end{bmatrix}\right) = r\left(\begin{bmatrix}1 & 1 & 1\\ 0 & 0 & z\\ 0 & 0 & 0\end{bmatrix}\right) \iff w = z \text{ or } w = \overline{z}.$$

Proof. Let $A_\psi = \begin{bmatrix}1 & 1 & 1\\ 0 & 0 & e^{i\psi}\\ 0 & 0 & 0\end{bmatrix}$, $f(\psi) = r(A_\psi)$, and

$$\mathcal{S} = \{v = \begin{bmatrix}a & be^{i\theta} & c\end{bmatrix}^t : a, b \geq 0,\ \theta \in [-\pi, \pi],\ |a|^2 + |b|^2 + |c|^2 = 1\}.$$

Then

$$\begin{aligned} f(\psi) &= \max_{v \in \mathcal{S}} |v^* A_\psi v| = \max_{v \in \mathcal{S}} |a^2 + abe^{i\theta} + c(a + be^{i(\psi - \theta)})| \\ &= \max_{v \in \mathcal{S}} |a^2 + abe^{i\theta}| + |c||a + be^{i(\psi-\theta)}|. \end{aligned}$$

For $\psi \in (-\pi, 0)$, the maximum is attained at some $\theta_0 \in [-\pi, 0]$ and for some $a_0, b_0, c_0 \neq 0$. (If $b_0 = 0$ or $c_0 = 0$, then $f(\psi) = \max\{a^2 + ab : a^2 + b^2 = 1\} = (1 + \sqrt{2})/2$. But $\lambda_{\max}((A_\psi^* + A_\psi)/2)$ is the largest root of $p(\lambda) = \lambda^3 - \lambda^2 - 3\lambda/4 + (1 - \cos\psi)/4$; since $p((1 + \sqrt{2})/2) < 0$ and $p(2) > 0$, $f(\psi) \geq \lambda_{\max} > (1 + \sqrt{2})/2$, giving a contradiction.) Thus for ϵ sufficiently small,

$$\begin{aligned} f(\psi) &= |a_0^2 + a_0 b_0 e^{i\theta_0}| + |c_0||a_0 + b_0 e^{i(\psi - \theta_0)}| \\ &< |a_0^2 + a_0 b_0 e^{i(\theta_0 + \epsilon)}| + |c_0||a_0 + b_0 e^{i((\psi+\epsilon) - (\theta_0 + \epsilon))}| \leq f(\psi + \epsilon) \end{aligned}$$

if $\theta_0 < 0$ and

$$\begin{aligned} f(\psi) &= a_0^2 + a_0 b_0 + |c_0||a_0 + b_0 e^{i\psi}| \\ &< a_0^2 + a_0 b_0 + |c_0||a_0 + b_0 e^{i(\psi+\epsilon)}| \le f(\psi+\epsilon) \end{aligned}$$

if $\theta_0 = 0$. Thus, f is strictly increasing on $[-\pi, 0]$. Since $f(-\psi) = f(\psi)$, the result follows. ■

2.2 Proof of Theorem 1.1

Assume that $r(A \circ B) = r(\phi(A) \circ \phi(B))$ for all $A, B \in M_n$. By Lemma 2.1, there are nonzero $\mu_{ij} \in \mathbb{C}$ such that

$$\{\phi(E_{ij}) : 1 \le i, j \le n\} = \{\mu_{ij} E_{ij} : 1 \le i, j \le n\}.$$

Step 1. There exists a permutation matrix P such that the mapping $X \mapsto P^t \phi(X) P$ will map E_{jj} to $\mu_{jj} E_{jj}$ with $|\mu_{jj}| = 1$ for all $j = 1, \ldots, n$.

Suppose, by way of contradiction, $\phi(E_{11}) = \mu_1 E_{rs}$ for some $r \ne s$. Then, since $1 = r(E_{11} \circ E_{11}) = r(\mu_1 E_{rs} \circ \mu_1 E_{rs})$, $|\mu_1| = \sqrt{2}$. Similarly, if $\phi(E_{12}) = \mu_2 E_{pq}$ then $|\mu_2| = 1$ if $p \ne q$ and $|\mu_2| = 1/\sqrt{2}$ if $p = q$. Let $X = E_{11} + E_{12}$. Since $r(X \circ E_{ij}) = r(\phi(X) \circ \phi(E_{ij}))$ for all i, j, we see that $\phi(X) = \xi_1 E_{rs} + \xi_2 E_{pq}$, where $|\xi_1| = \sqrt{2}$ and

$$|\xi_2| = \begin{cases} 1/\sqrt{2} & \text{if } p = q, \\ 1 & \text{if } p \ne q. \end{cases}$$

But then we obtain the contradiction

$$\frac{1+\sqrt{2}}{2} = r(X \circ X) = r(\xi_1^2 E_{rs} + \xi_2^2 E_{pq}) = \begin{cases} \frac{1+\sqrt{17}}{4} \text{ or } 1 \text{ if } p = q \\ 1, \sqrt{2}, \text{ or } \frac{\sqrt{5}}{2} \text{ if } p \ne q. \end{cases}$$

So, $\phi(E_{11}) = \mu_{jj} E_{jj}$ for some j. Since $1 = r(E_{11} \circ E_{11}) = r(\mu_{jj}^2 E_{jj})$, we see that $|\mu_{jj}| = 1$. A similar conclusion holds for $\phi(E_{kk})$ for all k, and our assertion follows.

Step 2. Without loss of generality, replace ϕ by the mapping $X \mapsto P^t \phi(X) P$, so $\phi(E_{jj}) = \mu_{jj} E_{jj}$ for all j. Moreover, for $p \ne q$, $\phi(E_{pq}) = \mu_{pq} E_{rs}$ for some $r \ne s$ and, since $1 = r(E_{pq} \circ E_{pq}) = r(\mu_{pq}^2 E_{rs})$, we have $|\mu_{pq}| = 1$. We show that $E_{rs} = E_{pq}$ or E_{qp}.

Let $X = E_{pp} + E_{pq}$. Since $r(X \circ E_{ij}) = r(\phi(X) \circ \phi(E_{ij})$ for all i, j, we have $\phi(X) = \xi_1 E_{pp} + \xi_2 E_{rs}$ where $|\xi_1| = |\xi_2| = 1$. Since

$$\frac{1+\sqrt{2}}{2} = r(X \circ X) = r(\xi_1^2 E_{pp} + \xi_2^2 E_{rs}),$$

E_{rs} must lie in the pth row or pth column. Similar consideration of $Y = E_{qq} + E_{pq}$ implies E_{rs} lies in the qth row or qth column, so $\phi(E_{pq}) = \mu_{pq} E_{pq}$ or $\mu_{pq} E_{qp}$ with $|\mu_{pq}| = 1$ as desired.

Step 3. We show that $\phi(J) = J \circ R$ where $R \circ R = \mu(\bar{x}_i x_j)$ with $|\mu| = |x_1| = \cdots = |x_n| = 1$.

Since $\{\phi(E_{ij}) : 1 \le i, j \le n\} = \{\mu_{ij} E_{ij} : 1 \le i, j \le n\}$ for some complex units μ_{ij}, and $r(E_{ij}) = r(J \circ E_{ij}) = r(\phi(J) \circ \phi(E_{ij}))$, we see that $\phi(J) = J \circ R$ for some unimodular R. Now,

$n = r(J \circ J) = r(\phi(J) \circ \phi(J)) = r(J \circ R \circ R)$. Applying Lemma 2.2 with $A = J$ and $U = R \circ R$, we get the desired conclusion.

Step 4. Interlude of four items.

Replace ϕ by the mapping $X \mapsto R^{(-1)} \circ \phi(X)$, where $R^{(-1)}$ is the Schur inverse of R having the (i,j)th entry equal to r_{ij}^{-1}. We may then assume that $\phi(J) = J$, and $r(\phi(A)) = r(\phi(A) \circ J) = r(A \circ J) = r(A)$ for all $A \in M_n$.

By using $|a_{ij}| = r(A \circ E_{ij}) = r(\phi(A) \circ \phi(E_{ij}))$, we have $\phi(A) = A^\sigma \circ U$ for some unimodular U. Here A^σ is the matrix obtained by performing some (perhaps none) local transpositions (swapping the a_{pq} and a_{qp} entries for some $p \neq q$).

Define an equivalence relation $A \sim B$ if $B = e^{i\theta} D^* A D$ for some diagonal unitary D and real number θ. Note that $A \sim B$ if and only if $B = A \circ (e^{i\theta} w w^*)$ for some real θ and unimodular vector w. Some simple properties of this relation are:

1. If $A \sim B$, then $\overline{A} \sim \overline{B}$ and $W(A) = W(B)$.
2. If $A_1 \sim B_1$ and $A_2 \sim B_2$, then $A_1 \circ A_2 \sim B_1 \circ B_2$.

Suppose B has positive entries in a principal 2×2 submatrix, and zero entries everywhere else. Then $r(B^t) = r(B) = r(\phi(B)) = r(B \circ U)$ or $r(B^t \circ U)$ for some unimodular U. By Lemma 2.2, $\phi(B) \sim B$ or B^t, depending on what ϕ does to the off-diagonal element. We shall repeatedly use this fact.

Step 5. We characterize the action of ϕ on all A supported on a particular 2×2 principal submatrix. To simplify notation, we let $n = 2$. If $\phi(E_{12}) = \mu E_{21}$, replace ϕ with $X \mapsto \phi(X)^t$ for this step. If A has two or more zero entries, then $\phi(A) \sim A \sim \overline{A}$, so suppose A has at most one zero entry. We consider three cases.

Case i: $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ for some $a, b, d \neq 0$. We can write $A \sim \begin{bmatrix} |a| & |b| \\ 0 & |d| e^{i\delta} \end{bmatrix}$ and $\phi(A) \sim \begin{bmatrix} |a| & |b| \\ 0 & |d| e^{i\theta} \end{bmatrix}$ where $\delta, \theta \in [-\pi, \pi]$. Let $B = \begin{bmatrix} 1/|a| & 1/|b| \\ 1 & 1/|d| \end{bmatrix}$, so $\phi(B) \sim B$. Then

$$r\left(\begin{bmatrix} 1 & 1 \\ 0 & e^{i\delta} \end{bmatrix}\right) = r(A \circ B) = r(\phi(A) \circ \phi(B)) = r\left(\begin{bmatrix} 1 & 1 \\ 0 & e^{i\theta} \end{bmatrix}\right).$$

By Lemma 2.3, $\theta = \pm\delta$, whence $\phi(A) \sim A$ or $\overline{A}$.

To show that either $\phi(A) \sim A$ for all $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$ or $\phi(A) \sim \overline{A}$ for all $A = \begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, let $C = \begin{bmatrix} 1 & 1 \\ 0 & i \end{bmatrix}$. By replacing ϕ with $\overline{\phi}$ if necessary, we may assume $\phi(C) \sim C$. Suppose $B = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ where $a, b, c, d > 0$. Either $\phi(B \circ C) \sim B \circ C$ or $\overline{B \circ C}$. In the latter case,

$$r\left(\begin{bmatrix} a & b \\ 0 & -d \end{bmatrix}\right) = r(C \circ (C \circ B)) = r(\phi(C) \circ \phi(C \circ B))$$

$$= r(C \circ \overline{C} \circ \overline{B}) = r\left(\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}\right),$$

contradicting Lemma 2.2. Hence $\phi(B \circ C) \sim B \circ C$ for any positive matrix B.

Define A, B as at the outset of this Case (i), and suppose $\phi(A) \sim \overline{A}$. Then

$$r\left(\begin{bmatrix} 1 & 1 \\ 0 & ie^{i\delta} \end{bmatrix}\right) = r(A \circ C \circ B) = r(\phi(A) \circ \phi(C \circ B))$$
$$= r(\overline{A} \circ C \circ B) = r\left(\begin{bmatrix} 1 & 1 \\ 0 & ie^{-i\delta} \end{bmatrix}\right),$$

whence, by Lemma 2.3, $\delta = 0$. Thus $\overline{A} \sim A$, so in fact $\phi(A) \sim A$ for all A of the form $\begin{bmatrix} a & b \\ 0 & d \end{bmatrix}$, as desired.

Case ii: $A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$ for some $a, b, c \neq 0$. We can write $A \sim \begin{bmatrix} |a| & |b| \\ |c|e^{i\gamma} & 0 \end{bmatrix}$ and $\phi(A) \sim \begin{bmatrix} |a| & |b| \\ |c|e^{i\theta} & 0 \end{bmatrix}$ where $\gamma, \theta \in [-\pi, \pi]$. Let $B = \begin{bmatrix} 1/|a| & 1/|b| \\ 1/|c| & 1 \end{bmatrix}$, so $\phi(B) \sim B$. Then

$$r\left(\begin{bmatrix} 1 & 1 \\ e^{i\gamma} & 0 \end{bmatrix}\right) = r(A \circ B) = r(\phi(A) \circ \phi(B)) = r\left(\begin{bmatrix} 1 & 1 \\ e^{i\theta} & 0 \end{bmatrix}\right).$$

By Lemma 2.4, $\theta = \pm\gamma$, whence $\phi(A) \sim A$ or $\overline{A}$. Using an argument similar to that in Case i, we may conclude that either $\phi(A) \sim A$ for all $A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$, or $\phi(A) \sim \overline{A}$ for all such A.

We now show that, assuming $\phi(A) \sim A$ for all A in Case i, $\phi(A) \sim A$ for all A in Case ii. By way of contradiction, suppose $\phi(A) \sim \overline{A}$ for all A in Case ii. Let $X = \begin{bmatrix} 1 & 1 \\ i & i \end{bmatrix}$, $Y = \begin{bmatrix} 1 & 1 \\ 0 & i \end{bmatrix}$, $Z = \begin{bmatrix} 1 & 1 \\ i & 0 \end{bmatrix}$. We write $\phi(X) \sim \begin{bmatrix} 1 & 1 \\ ie^{i\gamma} & ie^{i\delta} \end{bmatrix}$ for some $\gamma, \delta \in (-\pi, \pi]$. Then

$$r\left(\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\right) = r(X \circ Y) = r(\phi(X) \circ \phi(Y)) = r(\phi(X) \circ Y) = r\left(\begin{bmatrix} 1 & 1 \\ 0 & -e^{i\delta} \end{bmatrix}\right)$$

whence $\delta = 0$ by Lemma 2.3. Since $\phi(Z) \sim \overline{Z}$,

$$r\left(\begin{bmatrix} 1 & 1 \\ -1 & 0 \end{bmatrix}\right) = r(X \circ Z) = r(\phi(X) \circ \phi(Z)) = r(\phi(X) \circ \overline{Z}) = r\left(\begin{bmatrix} 1 & 1 \\ e^{i\gamma} & 0 \end{bmatrix}\right)$$

whence $\gamma = \pi$ by Lemma 2.4. But then

$$\sqrt{2} < r\left(\begin{bmatrix} 1 & 1 \\ i & i \end{bmatrix}\right) = r(X) = r(\phi(X)) = r\left(\begin{bmatrix} 1 & 1 \\ -i & i \end{bmatrix}\right) = \sqrt{2}$$

gives the desired contradiction. (Note $1 + i$ is an eigenvalue of X and X is not normal, so $1 + i$ lies in the interior of $W(X)$ and $r(X) > \sqrt{2}$. Meanwhile $\phi(X)$ is equivalent to $\sqrt{2}$ times a unitary matrix.) It follows that $\phi(A) \sim A$ for all $A = \begin{bmatrix} a & b \\ c & 0 \end{bmatrix}$, as desired.

Case iii: We still suppose $\phi(A) \sim A$ for all A in Case i. If A has a zero in the first row, we may use arguments similar to those in the first two cases to conclude that $\phi(A) \sim A$. Now suppose A has no zero entries. Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $Y = \begin{bmatrix} 1/a & 1/b \\ 0 & 1/d \end{bmatrix}$, and $Z = \begin{bmatrix} 1/a & 1/b \\ 1/c & 0 \end{bmatrix}$. We write $\phi(A) \sim \begin{bmatrix} a & b \\ ce^{i\gamma} & de^{i\delta} \end{bmatrix}$ where $\gamma, \delta \in [-\pi, \pi]$. Then, since $\phi(Y) \sim Y$,

$$r\left(\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\right) = r(A \circ Y) = r(\phi(A) \circ \phi(Y)) = r\left(\begin{bmatrix} 1 & 1 \\ 0 & e^{i\delta} \end{bmatrix}\right)$$

whence $\delta = 0$. Similar consideration of $r(A \circ Z)$ reveals $\gamma = 0$, and thus $\phi(A) \sim A$.

Step 6. We have shown that for all A supported on a given 2×2 principal submatrix, we have $\phi(A) \sim A^\tau$ where A^τ is one of: A, $\overline{A}$, A^t, or A^*. We shall show that ϕ has the same type of behavior on all 2×2 principal submatrices.

Case a: Conjugation.

Suppose now that $\phi(A) \sim A$ for A supported on the (p, q)-submatrix, and $\phi(A) \sim \overline{A}$ for A supported on the (r, s)-submatrix with $r = p$ or q. We show that this gives a contradiction. Without loss of generality, we take $p = 1$, $q = r = 2$, $s = 3$, and write all matrices as 3×3.

Let $w = \exp(i\pi/4)$, $A = \begin{bmatrix} w & 1 & 0 \\ 1 & w & 1 \\ 0 & 1 & w \end{bmatrix}$ and write $\phi(A) \sim \begin{bmatrix} w & 1 & 0 \\ a & wb & 1 \\ 0 & c & wd \end{bmatrix}$ where a, b, c, d are complex numbers of modulus one. Using Lemmas 2.3, 2.4, and

$$r(A \circ B) = r(\phi(A) \circ \phi(B)) = r(\phi(A) \circ B)$$

for $B = \begin{bmatrix} \overline{w} & 1 & 0 \\ 0 & \overline{w} & 0 \\ 0 & 0 & 0 \end{bmatrix}$ or $\begin{bmatrix} \overline{w} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, we have $a = b = 1$. Using Lemmas 2.3, 2.4, and

$$r(A \circ B) = r(\phi(A) \circ \phi(B)) = r(\phi(A) \circ \overline{B})$$

for $B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \overline{w} & 1 \\ 0 & 0 & \overline{w} \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 & 0 \\ 0 & \overline{w} & 1 \\ 0 & 1 & 0 \end{bmatrix}$, we have $d = 1$ and $c = w^4 = -1$.

Since A is a normal matrix with eigenvalues w and $w \pm \sqrt{2}$, we have $r(A) = |w + \sqrt{2}| = \sqrt{5}$. On the other hand, $\phi(A) - wI = UNU^*$ where

$$U = \begin{bmatrix} 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 1 & 0 \\ -1/\sqrt{2} & 0 & 1/\sqrt{2} \end{bmatrix} \text{ and } N = \begin{bmatrix} 0 & \sqrt{2} & 0 \\ 0 & 0 & \sqrt{2} \\ 0 & 0 & 0 \end{bmatrix}.$$

Since $W(N)$ is the unit disk, $r(\phi(A)) = 2 \neq r(A)$, giving the desired contradiction.

The same argument shows that we cannot have $\phi(A) \sim A^t$ for A supported on the (p, q)-submatrix, and $\phi(A) \sim A^*$ for A supported on the (r, s)-submatrix with $r = p$ or q.

Case b: Transposition.

Let $p < q < r$. We show that either $|\phi(E_{ij})| = E_{ij}$ for all $i < j$ in $\{p, q, r\}$, or $|\phi(E_{ij})| = E_{ji}$ for all $i < j$ in $\{p, q, r\}$. Suppose, by way of contradiction, this is not true. Without loss of generality, we take $p = 1$, $q = 2$, $r = 3$; write all matrices as 3×3; and assume that $|\phi(E_{12})| = E_{12}$, $|\phi(E_{13})| = E_{13}$, and $|\phi(E_{23})| = E_{32}$. By Case a, and by replacing ϕ with $\overline{\phi}$ if needed, we have $\phi(A) \sim A$ for all A supported on the $(1, 2)$- or $(1, 3)$-submatrix. Meanwhile $\phi(A) \sim A^t$ or A^* for all A supported on the $(2, 3)$-submatrix.

Let $w = e^{2\pi i/3}$, $A = \begin{bmatrix} 1 & 1 & 1 \\ w & \overline{w} & 1 \\ 1 & \overline{w} & w \end{bmatrix}$ and write $\phi(A) \sim \begin{bmatrix} 1 & 1 & 1 \\ aw & b\overline{w} & c\overline{w} \\ d & e & fw \end{bmatrix}$ for some unit scalars a, b, c, d, e, f. By using Lemmas 2.3, 2.4, and

$$r(A \circ B) = r(\phi(A) \circ \phi(B)) = r(\phi(A) \circ B)$$

for $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & w & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 1 & 0 \\ \overline{w} & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & \overline{w} \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, or $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 0 \end{bmatrix}$, we have $a = b = d = f = 1$, and $c = \overline{e}$ or $\overline{e}w$.

Let $A_\psi = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & e^{i\psi} \\ 0 & 0 & 0 \end{bmatrix}$ and $B_\psi = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & e^{i\psi} & 0 \end{bmatrix}$. A direct computation shows that $r(A_\psi) = r(B_\psi)$. If we write $\phi(A_0) \sim B_\theta$, Lemma 2.5 and $r(\phi(A_0)) = r(A_0)$ imply $\phi(A_0) \sim B_0$. Then Lemma 2.5 and $r(A \circ A_0) = r(\phi(A) \circ B_0)$ imply $e = 1$, so $c = 1$ or w. Hence

$$\phi(A) \sim \begin{bmatrix} 1 & 1 & 1 \\ w & \overline{w} & \overline{w} \\ 1 & 1 & w \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 1 & 1 & 1 \\ w & \overline{w} & 1 \\ 1 & 1 & w \end{bmatrix}.$$

But in the first case, $r(\phi(A)) \approx 1.65 < \sqrt{3}$ and in the second, $r(\phi(A)) \approx 1.87 > \sqrt{3}$. Since $r(A) = \sqrt{3}$, we have the desired contradiction.

Case c: Combining previous cases. By replacing ϕ with ϕ^t if necessary, we may assume $|\phi(E_{12})| = E_{12}$. Applying Case b with $p = 1$ and $q = 2$, we have $|\phi(E_{1r})| = E_{1r}$ for all r. Applying Case b with $p = 1$, we have $|\phi(E_{qr})| = E_{qr}$ for all $q < r$. It follows that given any 2×2 principal submatrix, either $\phi(A) \sim A$ or $\phi(A) \sim \overline{A}$ for all A supported on said submatrix.

By replacing ϕ with $\overline{\phi}$ if necessary, we may assume $\phi(A) \sim A$ for A supported on the $(1, 2)$-submatrix. By Case a, it follows that $\phi(A) \sim A$ for A supported on the $(1, p)$-submatrix for any p, and hence on the (p, q)-submatrix for any q as well. We conclude that $\phi(A) \sim A$ for any A supported on any 2×2 principal submatrix. (More generally, $\phi(A) = A \circ U$ for some unimodular U.)

Step 7. We show $\phi(A) \sim A$ for a special class of matrices (see Lemma 2.5) supported on a 3×3 principal submatrix.

We simplify notation by taking $n = 3$. Consider $A_\psi = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & e^{i\psi} \\ 0 & 0 & 0 \end{bmatrix}$, and write $\phi(A_\psi) \sim \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & e^{i\theta} \\ 0 & 0 & 0 \end{bmatrix}$ where $\psi, \theta \in [-\pi, \pi]$. By Lemma 2.5, $\theta = \pm\psi$, so $\phi(A_\psi) \sim A_\psi$ or $\overline{A_\psi}$.

To rule out $\phi(A_\psi) \sim \overline{A}_\psi$, we suppose, by way of contradiction, that $\phi(A_{\pi/4}) \sim \overline{A}_{\pi/4}$. Let $A = \begin{bmatrix} w & 1 & 1 \\ 1 & w & 1 \\ 1 & 1 & w \end{bmatrix}$ and write $\phi(A) \sim \begin{bmatrix} w & 1 & 1 \\ a & bw & c \\ d & e & fw \end{bmatrix}$. Here $w = \exp(i\pi/4)$ and a, b, c, d, e, f are unit scalars. By using Lemmas 2.3, 2.4, and

$$r(A \circ B) = r(\phi(A) \circ \phi(B)) = r(\phi(A) \circ B)$$

for $B = \begin{bmatrix} \overline{w} & 1 & 0 \\ 0 & \overline{w} & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \overline{w} & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} \overline{w} & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & \overline{w} \end{bmatrix}, \begin{bmatrix} \overline{w} & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$, or $\begin{bmatrix} 0 & 0 & 0 \\ 0 & \overline{w} & 1 \\ 0 & 1 & 0 \end{bmatrix}$, we have $a = b = d = f = 1$ and $e = \overline{c}$. By using Lemma 2.5,

$$r(A \circ A_{\pi/4}) = r(\phi(A) \circ \phi(A_{\pi/4})) = r(\phi(A) \circ \overline{A}_{\pi/4}),$$

and noting $\phi(A) \sim \begin{bmatrix} 1 & 1 & 1 \\ * & * & c/w \\ * & * & * \end{bmatrix}$, we have $c = w^2$.

Since A is a normal matrix with eigenvalues $w + 2$, $w - 1$, we have $r(A) = |2 + \exp(i\pi/4)|$. On the other hand, $\phi(A)$ is equivalent to a normal matrix with eigenvalues $w, w \pm \sqrt{3}$, so $r(\phi(A)) = |\sqrt{3} + \exp(i\pi/4)| < r(A)$, giving a contradiction. Thus $\phi(A_{\pi/4}) \sim A_{\pi/4}$.

If $\phi(A_\psi) \sim \overline{A}_\psi$, then, using the notation in the proof of Lemma 2.5,

$$f(\pi/4 + \psi) = r(A_{\pi/4} \circ A_\psi) = r(\phi(A_{\pi/4}) \circ \phi(A_\psi)) = f(\pi/4 - \psi).$$

By Lemma 2.5, $\psi = -\psi \mod 2\pi$, whence $A_\psi \sim \overline{A}_\psi$. Thus $\phi(A_\psi) \sim A_\psi$ for all ψ, as desired.

Step 8. We show that $\phi(A) \sim A$ for any A whose support is a 3×3 principal submatrix.

First let $B = \begin{bmatrix} a & b & c \\ * & * & d \\ * & * & * \end{bmatrix}$ have positive entries. Write $\phi(A_\psi \circ B) \sim \begin{bmatrix} a & b & c \\ 0 & 0 & de^{i\psi}e^{i\beta} \\ 0 & 0 & 0 \end{bmatrix}$. Then

$$\begin{aligned} r\left(\begin{bmatrix} a & b & c \\ 0 & 0 & d \\ 0 & 0 & 0 \end{bmatrix}\right) &= r(A_{-\psi} \circ A_\psi \circ B) \\ &= r(\phi(A_{-\psi}) \circ \phi(A_\psi \circ B)) = r\left(\begin{bmatrix} a & b & c \\ 0 & 0 & de^{i\beta} \\ 0 & 0 & 0 \end{bmatrix}\right) \end{aligned}$$

so $\beta = 0$ by Lemma 2.2, and $\phi(A_\psi \circ B) \sim A_\psi \circ B$.

Now suppose $A = [a_{ij}]$ has no zero entries. We write

$$A \sim \begin{bmatrix} |a_{11}| & |a_{12}| & |a_{13}| \\ |a_{21}|e^{i\alpha_{21}} & |a_{22}|e^{i\alpha_{22}} & |a_{23}|e^{i\alpha_{23}} \\ |a_{31}|e^{i\alpha_{31}} & |a_{32}|e^{i\alpha_{32}} & |a_{33}|e^{i\alpha_{33}} \end{bmatrix}$$

and

$$\phi(A) \sim \begin{bmatrix} |a_{11}| & |a_{12}| & |a_{13}| \\ |a_{21}|e^{i\alpha_{21}}e^{i\theta_{21}} & |a_{22}|e^{i\alpha_{22}}e^{i\theta_{22}} & |a_{23}|e^{i\alpha_{23}}e^{i\theta_{23}} \\ |a_{31}|e^{i\alpha_{31}}e^{i\theta_{31}} & |a_{32}|e^{i\alpha_{32}}e^{i\theta_{32}} & |a_{33}|e^{i\alpha_{33}}e^{i\theta_{33}} \end{bmatrix}$$

where $\alpha_{ij}, \theta_{ij} \in [-\pi, \pi]$. By using Lemmas 2.3, 2.4, and $r(A \circ B) = r(\phi(A) \circ B)$ where B is a matrix of the form in case (i) or (ii) of Step 5, and whose nonzero entries are reciprocals of those of A, we have $\theta_{ij} = 0$ for all $(i, j) \neq (2, 3)$ or $(3, 2)$, and $\theta_{23} = -\theta_{32}$. Now let $B = |A|^{(-1)}$ (absolute value and inverse operations are entry-wise). Using the notation in the proof of Lemma 2.5,

$$\begin{aligned} f(0) &= r(A \circ B \circ A_{-\alpha_{23}}) = r(\phi(A) \circ \phi(B \circ A_{-\alpha_{23}})) \\ &= r(\phi(A) \circ B \circ A_{-\alpha_{23}}) = f(\theta_{23}), \end{aligned}$$

so $\theta_{23} = 0$. Our assertion follows.

Step 9. We consider $n \times n$ matrices.

First consider an $n \times n$ matrix A such that $A_{ij} \neq 0 \iff i, j \in I = \{i_1, \ldots, i_k\}$ where $1 \leq i_1 < \cdots < i_k \leq n$ and $k \geq 3$. Let $A = [a_{ij}]$ and write $\phi(A) \sim [a_{ij}e^{i\theta_{ij}}]$ where $\theta_{ij} \in [-\pi, \pi]$ for all i, j and $\theta_{i_1 j} = 0$ for all $j \in I$. Let $i_1 < p < q$ with $p, q \in I$. Let B be supported on the 3×3 principal submatrix on (i_1, p, q) with nonzero entries $1/a_{ij}$. Since $r(A \circ B) = r(\phi(A) \circ B)$, Lemma 2.2 implies

$$\theta_{pq} = \theta_{qp} = \theta_{pp} = \theta_{qq} = \theta_{pi_1} = \theta_{qi_1} = 0.$$

Thus $\phi(A) \sim A$ if A's support is a principal submatrix. In particular, $\phi(A) \sim A$ if A has no zero entries.

Let A be an $n \times n$ matrix such that $|A| + |A|^t$ is irreducible. Write $\phi(A) = A \circ R$ where R is a unimodular matrix. Define $B_{ij} = |A_{ij}|/A_{ij}$ if $A_{ij} \neq 0$ and $B_{ij} = 1$ otherwise. Then

$$r(|A|) = r(A \circ B) = r(\phi(A) \circ \phi(B)) = r(\phi(A) \circ B) = r(|A| \circ R)$$

so it follows by Lemma 2.2 that $\phi(A) \sim A$.

Finally, in the most general case, let Q be a permutation such that $Q^tAQ = A_1 \oplus \cdots \oplus A_k$, where each A_j is indecomposable and so $|A_j| + |A_j|^t$ is irreducible. Without loss of generality, we take $Q = I$ to simplify notation. Write $\phi(A) = A \circ R$ where R is a unimodular matrix, and let R_j be the submatrix of R corresponding to A_j. Define B_j to be the $n \times n$ matrix whose support is the principal submatrix underlying A_j, and whose nonzero entries are either $|(A_j)_{pq}|/(A_j)_{pq}$, if $(A_j)_{pq} \neq 0$, or 1. Then for each j

$$r(|A_j|) = r(A \circ B_j) = r(\phi(A) \circ \phi(B_j)) = r(A \circ B_j \circ R) = r(|A_j| \circ R_j),$$

so, by Lemma 2.2, we may (by redefining those entries of R_j corresponding to zero entries for A_j if needed) assume $R_j = \lambda_j w_j w_j^*$ for some unit scalar λ_j and unimodular vector w_j. Let $D_j = \mathrm{diag}\,(w_j)$. Then

$$\phi(A) = A \circ R = \oplus_{j=1}^k A_j \circ R_j = \oplus_{j=1}^k \lambda_j D_j A_j D_j^* = DAE$$

where $D = \oplus_{i=1}^k D_j$ and $E = \oplus_{i=1}^k \lambda_j D_j^*$. ■

2.3 Proof of Theorem 1.2

Assume $W(A \circ B) = W(\phi(A) \circ \phi(B))$ for all $A, B \in M_n$. Thus $r(A \circ B) = r(\phi(A) \circ \phi(B))$ and so ϕ has one of the forms in Theorem 1.1. By replacing ϕ with $X \mapsto R^{(-1)} \circ \phi(PXP^t)$ or $X \mapsto R^{(-1)} \circ \phi(PX^tP^t)$, we may assume that $\phi(X) = D_X X E_X$ or $D_X \overline{X} E_X$ where D_X, E_X are diagonal unitaries such that $D_X E_X$ commutes with X. Note that if X is indecomposable, then $D_X E_X = \lambda I$ and so $\phi(X) = \lambda D_X X D_X^*$ or $\phi(X) = \lambda D_X \overline{X} D_X^*$ for some unit scalar λ.

Step 1. Fixing J.

We have $\phi(J) = \lambda D_J J D_J^*$. Replacing ϕ with $X \mapsto D_J^* \phi(X) D_J$, we may assume $\phi(J) = \lambda J$. Since $[0, n] = W(J \circ J) = W(\phi(J) \circ \phi(J)) = \lambda^2 W(J)$, we have $\lambda = \pm 1$. Replacing ϕ with $X \mapsto \lambda J \circ \phi(X)$, we may assume $\phi(J) = J$, and $W(\phi(A)) = W(\phi(A) \circ J) = W(A \circ J) = W(A)$ for all A. Note that ϕ still has one of the forms $\phi(X) = D_X X E_X$ or $D_X \overline{X} E_X$ for all X.

Step 2. Conjugation.

Let $X = \left(i \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} + 2 \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \right) \oplus 0_{n-2}$, so X is an indecomposable normal matrix such that $W(X)$ is either the line segment joining i and 2 (if $n = 2$) or the triangle with vertices at $0, i, 2$. Since $W(\lambda D_X \overline{X} D_X^*) = \lambda W(\overline{X}) \neq W(X)$ for any complex unit λ, we must have $\phi(X) = D_X X E_X$ for all X.

For the next 3 steps, we assume A is an indecomposable matrix, so $\phi(A) = \lambda D_A A D_A^*$ and $W(A) = W(\phi(A)) = \lambda W(A)$ for some complex unit λ. We shall show that we can take $\lambda = 1$ in each case (i.e., $\phi(A) \approx A$, where we define an equivalence relation $A \approx B$ if $A = DBD^*$ for some diagonal unitary D.)

Step 3. Nonnegative indecomposable matrices.

If A is a nonnegative indecomposable matrix, then $H = (A + A^t)/2$ is irreducible and has a unique positive unit eigenvector x such that $x^t A x = r(A)$. Since $W(A) = \lambda W(A)$, there is a unit vector v such that $v^* A v = \lambda x^t A x$. Let D be a diagonal unitary such that $D|v| = v$. Following the proof of Lemma 2.2 (set $\tilde{A} = A$), we have $|v| = x$ and $D^* A D = \mu A$ for some complex unit μ. Then $\lambda x^t A x = v^* A v = x^t D^* A D x = \mu x^t A x$, so $\mu = \lambda$. Thus $\phi(A) \approx \lambda A \approx A$ as desired.

Step 4. Full matrices.

Suppose all of A's entries are nonzero. Define a positive matrix B by $B_{11} = n|A_{11}|^{-1}$ and $B_{ij} = ((n^2 - 1)|A_{ij}|)^{-1}$ for all other (i, j)-entries. We have

$$\begin{aligned} W(A \circ B) &= W(\phi(A) \circ \phi(B)) = W(\lambda D_A A D_A^* \circ D_B B D_B^*) \\ &= \lambda W((D_A D_B)(A \circ B)(D_A D_B)^*) = \lambda W(A \circ B). \end{aligned} \tag{2.1}$$

Let $C = A \circ B$, and write $C_{ij} = |C_{ij}|e^{i\theta_{ij}}$. Let x be a unit vector. Then

$$|x^*Cx - e_1^tCe_1| \leq |ne^{i\theta_{11}}|x_1|^2 - ne^{i\theta_{11}}| + |\sum_{(i,j)\neq(1,1)} \frac{1}{n^2-1}\overline{x}_i x_j e^{i\theta_{ij}}| \leq n+1.$$

Thus $W(C)$ lies inside a circle of radius $n+1$ about C_{11}. But if $\lambda \neq 1$, then $W(C) = \lambda W(C)$ implies that $\lambda^k C_{11} \in W(C)$ for all k. Choose k such that $2\pi/3 \leq \arg \lambda^k \leq 4\pi/3$. Then

$$n+1 \geq |\lambda^k C_{11} - C_{11}| \geq |e^{2\pi i/3}C_{11} - C_{11}| = \sqrt{3}n$$

contradicts $n \geq 2$, so $\lambda = 1$.

Step 5. Arbitrary indecomposable matrices.

Let A be an arbitrary indecomposable matrix. Define a full matrix B by $B_{ij} = |A_{ij}|/A_{ij}$ if $A_{ij} \neq 0$, and $B_{ij} = 1$ otherwise. Using (2.1) we have $W(|A|) = \lambda W(|A|)$. By step 3, $|A| = \lambda D|A|D^*$ for some diagonal unitary D. Write $|A| = A \circ R$ for some unimodular R. Then $A \circ R = \lambda D(A \circ R)D^* = \lambda(DAD^*) \circ R$, so $A = \lambda DAD^*$. Then $\phi(A) \approx \lambda A \approx A$ as desired.

Step 6. General matrices.

Let Q be a permutation such that $Q^tAQ = A_1 \oplus \cdots \oplus A_k$ where each A_j is indecomposable. Without loss of generality, we take $Q = I$ to simplify notation. We have $\phi(A) = \lambda_1 D_1 A_1 D_1^* \oplus \cdots \oplus \lambda_k D_k A_k D_k^*$ for some complex units λ_j and diagonal unitaries D_j. The arguments in Lemma 2.2 and the preceding three steps readily apply to matrices of the form $A = A_1 \oplus 0$ with A_1 indecomposable, in which case it follows $\phi(A_1 \oplus 0) = D_1 A_1 D_1^* \oplus 0$ for some diagonal unitary D_1. Let $B = B_1 \oplus 0 \oplus \cdots \oplus 0$, where B_1 is a matrix of the same size as A_1 such that $A_1 \circ B_1 = |A_1|$. Since $\phi(B) = DBD^*$ for some diagonal unitary, we have

$$W(|A_1| \oplus 0) = W(A \circ B) = W(\phi(A) \circ \phi(B)) = \lambda_1 W(|A_1| \oplus 0).$$

The arguments in steps 3 and 5 imply $|A_1| \oplus 0 \approx \lambda_1(|A_1| \oplus 0)$, so $|A_1| \approx \lambda_1 |A_1|$ and thus $A_1 \approx \lambda_1 A_1$. Similarly $A_j \approx \lambda_j A_j$ for all j, and so $\phi(A) \approx A$ as desired.

∎

3 Proofs for Hermitian matrices

3.1 Auxiliary results

Lemma 3.1. *Suppose $\mathcal{S} \subseteq H_n$ has $n(n+1)/2$ nonzero elements such that $X \circ Y = 0$ for any $X \neq Y \in \mathcal{S}$. Then there are nonzero scalars $\mu_{ij} \in \mathbb{C}$ such that*

$$\mathcal{S} = \{\mu_{ij}E_{ij} + \bar{\mu}_{ij}E_{ji} : 1 \leq i \leq j \leq n\}.$$

Lemma 3.2. *Let $f(t) = r(A_t)$ where*

$$A_t = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 0 & e^{it} \\ 1 & e^{-it} & 0 \end{bmatrix}.$$

Then $f(s) = f(t)$ for $s, t \in [-\pi, \pi]$ if and only if $s = \pm t$.

Proof. Since $\det(A_t) = 2\cos t - 2 < 0$ for $t \in (0, \pi]$, we see that A_t has eigenvalues $\lambda_1(t) \geq \lambda_2(t) > 0 > \lambda_3(t)$. Since $\det(A_t - zI) = -z^3 + 2z^2 + 3z + 2\cos t - 2$, $\lambda_1(t)$ (respectively, $|\lambda_3(t)|$) clearly decreases (respectively, increases) as t increases from 0 to π. Since $\lambda_1(\pi) = (1+\sqrt{17})/2 > |(1-\sqrt{17})/2| = |\lambda_3(\pi)|$, it follows that $f(t) = \lambda_1(t)$ and hence strictly decreases on $[0, \pi]$. Since f is even, the result follows. ∎

3.2 Proof of Theorem 1.1

Assume that $r(\phi(A) \circ \phi(B)) = r(A \circ B)$ for all $A, B \in H_n$. Define $A \sim B$ if $A = \pm D^* AD$ for some diagonal unitary D.

Step 1. There is a permutation P and complex units μ_{ij} with $\mu_{11}, \ldots, \mu_{nn} \in \{1, -1\}$ such that

$$\phi(E_{ij} + E_{ji}) = P^t(\mu_{ij} E_{ij} + \bar{\mu}_{ij} E_{ji})P$$

for all $1 \leq i \leq j \leq n$.

Consider

$$\mathcal{S} = \{E_{11}, \ldots, E_{nn}\} \cup \{E_{ij} + E_{ji} : 1 \leq i < j \leq n\}.$$

Since $0 = r(X \circ Y) = r(\phi(X) \circ \phi(Y))$ for all $X \neq Y \in \mathcal{S}$ and $1 = r(X \circ X) = r(\phi(X) \circ \phi(X))$ for all $X \in \mathcal{S}$, Lemma 3.1 implies that the image of $\mathcal{S}$ under ϕ is

$$\{\mu_{11} E_1, \ldots, \mu_{nn} E_{nn}\} \cup \{\mu_{ij} E_{ij} + \bar{\mu}_{ij} E_{ji} : 1 \leq i < j \leq n\}$$

where $|\mu_{ij}| = 1$ and $\mu_{ii} = \pm 1$.

Suppose, by way of contradiction, that $\phi(E_{12} + E_{21}) = \pm E_{ii}$ for some i. If $n = 2$, then $r(X \circ I) = r(\phi(X) \circ \phi(I))$ for all $X \in \mathcal{S}$ shows that $\phi(I)$ has unit entries except for one zero diagonal entry. Then $1 = r(I \circ I) = r(\phi(I) \circ \phi(I)) = (1+\sqrt{5})/2$, a contradiction.

If $n > 2$, let $Y = E_{12} + E_{21} + E_{23} + E_{32}$. Then, no matter what $\phi(E_{23} + E_{32})$ is, $\sqrt{2} \neq r(\phi(Y) \circ \phi(Y)) = r(Y \circ Y) = \sqrt{2}$, a contradiction. Thus, after applying a permutation similarity, we may assume that $\phi(E_{jj}) = \pm E_{jj}$. Now, let $Y = E_{ii} + E_{jj} + E_{ij} + E_{ji}$. Since $2 = r(Y \circ Y) = r(\phi(Y) \circ \phi(Y))$, we must have $\phi(E_{ij} + E_{ji}) = \mu_{ij} E_{ij} + \bar{\mu}_{ij} E_{ji}$ for some unit μ_{ij}, as desired.

Step 2. The conclusion of the theorem holds for irreducible nonnegative matrices and matrices with nonzero support on a 2×2 principal submatrix.

By Lemma 2.2 and $r(J) = r(J \circ J) = r(\phi(J) \circ \phi(J))$, we have $\phi(J) = J \circ R$ where R is a unimodular hermitian and $R \circ R = (\bar{x}_i x_j)$ with $|x_1| = \cdots = |x_n| = 1$. By replacing ϕ with $A \mapsto \phi(J)^{(-1)} \circ \phi(A)$, we may assume $\phi(J) = J$ and consequently, $r(A) = r(\phi(A))$ for all A. By Lemma 2.2, $\phi(A) \sim A$ for all irreducible nonnegative A.

Suppose A has nonzero support on a 2×2 principal submatrix. We can write $A \sim \begin{bmatrix} a & b \\ b & d \end{bmatrix}$ where $a, b > 0$, and $\phi(A) \sim \begin{bmatrix} a & b \\ b & \mu d \end{bmatrix}$ where $\mu = \pm 1$. Since $r(\phi(A)) = r(A)$, we must have $\mu = 1$ and $\phi(A) \sim A$ (apply Lemma 2.2).

Step 3. The conclusion of the theorem holds for matrices with nonzero support in a 3×3 principal submatrix.

Suppose A has nonzero support on a given 3×3 principal submatrix. We can write

$$A \sim \begin{bmatrix} a & b & c \\ b & d & re^{it} \\ c & re^{-it} & f \end{bmatrix} \quad \text{and } \phi(A) \sim \begin{bmatrix} a & b & c \\ b & \alpha d & \mu r \\ c & \bar{\mu} r & \beta f \end{bmatrix}$$

where $a, b, c, r > 0$, $t \in \mathbb{R}$, $\alpha, \beta \in \{1, -1\}$, and $|\mu| = 1$. Setting $X = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$, or $\begin{bmatrix} 2/a & 1/b & 1/c \\ 1/b & 0 & 1/r \\ 1/c & 1/r & 0 \end{bmatrix}$, and using $r(X \circ A) = r(\phi(X) \circ \phi(A)) = r(X \circ \phi(A))$, Lemma 2.2, and Lemma 3.2 gives $\alpha = \beta = 1$ and $\mu = e^{it}$ or e^{-it}. Hence $\phi(A) \sim A$ or $\bar{A}$.

Let $C = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 0 & i \\ 1 & -i & 0 \end{bmatrix}$. By replacing ϕ with $\bar{\phi}$ if needed, we may assume $\phi(C) \sim C$. Let B be any matrix with positive entries. If $\phi(B \circ C) \sim \overline{B \circ C}$, then

$$r(C \circ B \circ C) = r(\phi(C) \circ \phi(B \circ C)) = r(C \circ \bar{B} \circ \bar{C}) = r(|C| \circ B)$$

contradicts Lemma 2.2. Thus $\phi(B \circ C) \sim B \circ C$ for any positive matrix B.

Now if $\phi(A) \sim \bar{A}$ then, writing $B = |A|^{(-1)}$ and using the notation in Lemma 3.2,

$$f(t + \pi/2) = r(A \circ B \circ C) = r(\phi(A) \circ \phi(B \circ C)) = r(\bar{A} \circ B \circ C) = f(-t + \pi/2),$$

whence $e^{it} \in \mathbb{R}$ and $\phi(A) \sim A \sim \bar{A}$.

Step 4. We have shown that for all A supported on a given 3×3 principal submatrix, either $\phi(A) \sim A$ or $\phi(A) \sim \bar{A}$. Suppose, by way of contradiction, that $\phi(A) \sim A$ for A supported on one 3×3 submatrix and $\phi(A) \sim \bar{A}$ for A supported on a different 3×3 submatrix. Without loss of generality, we may suppose $\phi(A) \sim A$ for all A supported on the $(1,2,3)$- or $(1,3,4)$-submatrix and $\phi(A) \sim \bar{A}$ for all A supported on the $(2,3,4)$-submatrix. We write

$$A = \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & i & 1 \\ 1 & -i & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, \quad \phi(A) \sim \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & \alpha & \mu & 1 \\ 1 & \bar{\mu} & \beta & \nu \\ 0 & 1 & \bar{\nu} & \gamma \end{bmatrix}$$

where $\alpha, \beta, \gamma \in \{\pm 1\}$ and $|\mu| = |\nu| = 1$. Using $r(A \circ X) = r(\phi(A) \circ \phi(X)) = r(\phi(A) \circ X)$ and Lemma 2.2 when X is a $(0,1)$-matrix with nonzero support on a 2×2 principal submatrix implies $\alpha = \beta = \gamma = 1$. When X is the Schur inverse of the leading 3×3 principal submatrix of A, we get $\mu = i$. When X is the Schur inverse of the $(1,3,4)$-submatrix of A, we get $\nu = 1$. When X is the Schur inverse of the $(2,3,4)$-submatrix of A, we have $r(A \circ X) = r(\phi(A) \circ \phi(X)) = r(\phi(A) \circ \bar{X})$, contradicting Lemma 2.2.

Thus, by replacing ϕ with $\bar{\phi}$ if needed, we may assume that $\phi(A) \sim A$ for all A with support in the leading 3×3 principal submatrix, and hence $\phi(A) \sim A$ for all A supported on any 3×3 principal submatrix.

The rest of the proof is exactly the same as step 9 for complex matrices, and we conclude $\phi(A) \sim A$ for all $A \in H_n$.

■

The proof of Theorem 1.2 follows the analogous proof in the complex case.

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