On numerical ranges and roots

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Abstract Existence of the fractional powers is established in Banach algebra setting, in terms of the numerical ranges of elements involved. The behavior of the spectra and (for Hermitian *-algebras satisfying some additional hypotheses) the *-numerical range under taking these powers also is investigated.

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0 Introduction

Let $\mathcal{H}$ be a Hilbert space with the inner product $(\cdot,\cdot)$, and let $\mathcal{L}(\mathcal{H})$ be the algebra of bounded linear operators on $\mathcal{H}$. For $A \in \mathcal{L}(\mathcal{H})$, let

$$W(A) = \{(Ax, x) \,:\, x \in \mathcal{H}, (x, x) = 1\}$$

be the numerical range of A.

The following result was proved (in a slightly different form) in [15], and extended in [12], see also [11], to certain classes of unbounded operators:

Theorem 0.1 *If $A \in \mathcal{L}(\mathcal{H})$ and $W(A)$ does not contain any negative real numbers, then for every positive integer p there exists a unique $B_p \in \mathcal{L}(\mathcal{H})$ such that $B_p^p = A$ and*

$$W(B) \subseteq \{z = re^{i\alpha} \in \mathbb{C} \,:\, r \geq 0, \quad |\alpha| \leq \frac{\pi}{p}\}.$$

Using techniques of linear algebra, Theorem 0.1 (for $p = 2$), was proved for finite dimensional $\mathcal{H}$ in [9], [10], [8], [16].

Taking cue from Theorem 0.1, in this note we prove results concerning existence and uniqueness of roots of elements in a Banach algebra, under suitable hypotheses on numerical ranges. The proofs of our main results – Theorems 1.2 and 2.8, – make heavy use of the techniques from [15]. Since the latter paper is available in Russian only, we decided to include at least some details rather than merely give a reference. We hope that readers will find that convenient.

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1 Banach algebras setting

All Banach algebras will be assumed complex and unital with the unit e such that $\|e\| = 1$. Let $\mathcal{A}$ be a Banach algebra. For every element $a \in \mathcal{A}$, define the *Banach algebra numerical range* $V(a)$ as follows:

$$V(a) = \{f(a)\colon\, f \in S\} \subseteq \mathbb{C},$$

where S is the set of bounded linear functionals f on $\mathcal{A}$ such that $f(e) = ||f|| = 1$ (such functionals are called *states* of $\mathcal{A}$). This notion is standard, see, e.g., [1], [2] and references there. Numerical ranges of Banach algebra elements come up in a variety of settings, see, e.g., [20] for results concerning nearly Hermitian elements.

We begin with some elementary properties of $V(a)$.

Proposition 1.1 *(1) The set $V(a)$ is closed, convex, and bounded.*

(2) $\sigma(a) \subseteq V(a)$.

(3) If $\lambda \in \mathbb{C} \setminus V(a)$, then $\|(\lambda e - a)^{-1}\| \leq d^{-1}$, where d is the distance from λ to $V(a)$.

(4) If $\mathcal{A} = \mathcal{L}(\mathcal{H})$, then $V(a)$ is the closure of $W(a)$, for every $a \in \mathcal{L}(\mathcal{H})$.

Properties (1),(2), and (4) are proved in [1]; (3) is proved in [21].

The following result was proved in [15] for the case $\mathcal{A} = \mathcal{L}(\mathcal{H})$.

Theorem 1.2 *Let $\mathcal{A}$ be a Banach algebra, and let $a \in \mathcal{A}$ be such that*

$$V(a) \text{ does not contain any negative real numbers.} \tag{1.1}$$

Let $\mathcal{A}(a)$ be the closed unital subalgebra of $\mathcal{A}$ generated by a and e. Then for every ω, $0 < \omega < 1$, there exists $b_\omega \in \mathcal{A}(a)$ such that

$$b_{\omega_1} b_{\omega_2} = \begin{cases} b_{\omega_1+\omega_2} & \text{if } \omega_1 + \omega_2 < 1 \\ a & \text{if } \omega_1 + \omega_2 = 1 \\ ab_{\omega_1+\omega_2-1} & \text{if } \omega_1 + \omega_2 > 1 \end{cases} \tag{1.2}$$

and

$$\sigma(b_\omega) \subseteq \{re^{i\theta} \in \mathbb{C}\colon\, r \geq 0,\ |\theta| < \omega\pi\}.$$

If in addition a is invertible and $\omega = \frac{1}{m}$ is the reciprocal of an integer, then b_ω with the above properties in unique.

Proof. We may assume, using compactness and convexity of $V(a)$ and the hypothesis (1.1), that there exist α, $0 < \alpha < \pi$, and $R > 0$, and δ, $0 < \delta < \min\{\alpha, R\}$, such that $V(a)$ is contained in the set

$$\{z = re^{i\theta} \in \mathbb{C}\colon\, 0 \leq r \leq R - \delta, -\alpha + \delta \leq \theta \leq \alpha - \delta\}.$$

(If it happens that the real line is tangential to $V(a)$ at zero, we replace a with $e^{i\tau}a$ for some τ sufficiently close to zero; then b_p is replaced with $e^{i\tau\omega}a$.) Let Γ be the positively oriented contour composed of a part of the circle of radius R centered at zero, and of two symmetric line segments that connect zero with the circle, as follows:

$$\Gamma = \{z = Re^{i\theta} \in \mathbb{C}\colon\ -\alpha \leq \theta \leq \alpha\}$$

$$\cup \{z = re^{i\alpha} \in \mathbb{C}\colon\ 0 \leq r \leq R\} \cup \{z = re^{-i\alpha} \in \mathbb{C}\colon\ 0 \leq r \leq R\}.$$

If μ, ν are positive real numbers smaller then R, we let $\Gamma_{\mu,\nu}$ be the curve obtained from Γ by cutting out segments with endpoint zero of lengths μ and ν from the two symmetric line segments:

$$\Gamma_{\mu,\nu} = \{z = Re^{i\theta} \in \mathbb{C}\colon\ -\alpha \leq \theta \leq \alpha\}$$

$$\cup \{z = re^{i\alpha} \in \mathbb{C}\colon\ \mu \leq r \leq R\} \cup \{z = re^{-i\alpha} \in \mathbb{C}\colon\ \nu \leq r \leq R\}.$$

Let $\varepsilon_0 > 0$ be so small that the spectral radius of $a+\varepsilon e$ is smaller than R, for every $\varepsilon \in [0, \varepsilon_0]$.

Consider the following curve integrals:

$$I(\mu, \nu, \varepsilon) := \frac{1}{2\pi i} \int_{\Gamma_{\mu,\nu}} (\lambda)^{\omega} (\lambda e - (a + \varepsilon e))^{-1} d\lambda \in \mathcal{A},$$

where $0 \leq \varepsilon \leq \varepsilon_0$, and where $(\lambda)^{\omega}$ is the analytic branch of the ω-th power function defined by the property that $(\lambda)^{\omega} > 0$ for $\lambda > 0$. Define

$$a_{\varepsilon,\omega} := \lim_{\mu,\nu\to 0} I(\mu, \nu, \varepsilon) \in \mathcal{A}.$$

For $0 < \varepsilon \leq \varepsilon_0$ the limits $a_{\varepsilon,\omega}$ exist, and by functional calculus we have

$$a_{\varepsilon,\omega_1} a_{\varepsilon,\omega_2} = \begin{cases} a_{\varepsilon,\omega_1+\omega_2} & \text{if } \omega_1 + \omega_2 < 1 \\ a + \varepsilon e & \text{if } \omega_1 + \omega_2 = 1 \\ (a + \varepsilon e) a_{\varepsilon,\omega_1+\omega_2-1} & \text{if } \omega_1 + \omega_2 > 1 \end{cases} \tag{1.3}$$

and

$$\sigma(a_{\varepsilon,\omega}) \subseteq \{z = re^{i\theta} \in \mathbb{C}\colon\ r > 0, -\omega\pi < \theta < \omega\pi\}. \tag{1.4}$$

To include also the case when $\varepsilon = 0$, we argue as follows. Fix $\mu, \nu, \mu', \nu' \in (0, R)$, and suppose for simplicity of notation that $\mu' \leq \mu$, $\nu' \leq \nu$. Then

$$2\pi \|I(\mu, \nu, \varepsilon) - I(\mu', \nu', \varepsilon)\| \leq \int |(\lambda)^{\omega}| \|(\lambda e - (a + \varepsilon e))^{-1}\| \, |d\lambda|, \tag{1.5}$$

where the integral is taken over two line segments

$$\{z = re^{i\alpha} \in \mathbb{C}\colon\ \mu' \leq r \leq \mu\} \cup \{z = re^{-i\alpha} \in \mathbb{C}\colon\ \nu' \leq r \leq \nu\}. \tag{1.6}$$

By Proposition 1.1(3), for $\lambda = re^{\pm i\alpha}$, $r > 0$ we have

$$\|(\lambda e - (a + \varepsilon e))^{-1}\| \le \frac{1}{r \sin \delta},$$

and therefore the right hand side of (1.5) does not exceed

$$\int_{\mu' \le r \le \mu} r^{\omega} \frac{1}{r \sin \delta} dr + \int_{\nu' \le r \le \nu} r^{\omega} \frac{1}{r \sin \delta} dr = \frac{1}{\omega \sin \delta} \left((\mu')^{\omega} - (\mu)^{\omega} + (\nu')^{\omega} - (\nu)^{\omega} \right),$$

which tends to zero as $\mu', \mu, \nu', \nu \to 0$. Thus, $a_{\varepsilon,\omega}$, and in particular $a_{0,\omega}$, converges in $\mathcal{A}$. Moreover, the convergence

$$\lim_{\mu,\nu \to 0} I(\mu, \nu, \varepsilon)$$

is uniform in $\varepsilon \in [0, \varepsilon_0]$. Also, for every fixed μ, ν $(0 < \mu, \nu < R)$, the convergence

$$\lim_{\varepsilon \to 0} \left[(\lambda)^{\omega} (\lambda e - (a + \varepsilon e))^{-1} \right] = (\lambda)^{\omega} (\lambda e - a)^{-1}$$

is uniform on $\Gamma_{\mu,\nu}$, because the spectra $\sigma(a + \varepsilon e)$, $0 \le \varepsilon \le \varepsilon_0$, are uniformly separated from $\Gamma_{\mu,\nu}$. By a well known theorem on integrals depending on a parameter (see [6], for example), we have

$$\begin{aligned}
\lim_{\varepsilon \to 0} a_{\varepsilon,\omega} &= \lim_{\varepsilon \to 0} \lim_{\mu,\nu \to 0} I(\mu, \nu, \varepsilon) \\
&= \frac{1}{2\pi} \lim_{\mu,\nu \to 0} \int_{\Gamma_{\mu,\nu}} \left[\lim_{\varepsilon \to 0} (\lambda)^{\omega} (\lambda e - (a + \varepsilon e))^{-1} \right] d\lambda \\
&= \frac{1}{2\pi} \lim_{\mu,\nu \to 0} \int_{\Gamma_{\mu,\nu}} (\lambda)^{\omega} (\lambda e - a)^{-1} d\lambda \\
&= a_{0,\omega}.
\end{aligned}$$

Passing to the limit when $\varepsilon \to 0$ in (1.3), we obtain equalities (1.2), with $b_{\omega} = a_{0,\omega}$.

The proof above shows that $a_{\varepsilon,\omega} \in \mathcal{A}(a)$ for $0 \le \varepsilon \le \varepsilon_0$. Since the set $\{z \in \mathbb{C}\colon\ z = re^{i\alpha},\ r > 0,\ |\alpha| < \omega\pi\}$ is convex, in view of (1.4), also

$$\sigma_0(a_{\varepsilon,\omega}) \subset \{z \in \mathbb{C}\colon\ z = re^{i\alpha},\ r > 0,\ |\alpha| < \omega\pi\}, \tag{1.7}$$

where $\sigma_0(x)$ is the spectrum of $x \in \mathcal{A}(a)$ with respect to the algebra $\mathcal{A}(a)$. Let X be the compact Hausdorff space of maximal ideals of $\mathcal{A}(a)$, and let $\widehat{x} \in C(X)$, the Banach space of continuous complex functions on X with the maximum modulus norm, be the Gelfand transform of $x \in \mathcal{A}(a)$. Since the Gelfand transform is continuous (for this and other properties of Gelfand transform used here, see, e.g., [4], or [17, Theorem 3.1.5]), we have

$$\lim_{\varepsilon \to 0} \widehat{a_{\varepsilon,\omega}} = \widehat{a_{0,\omega}}. \tag{1.8}$$

Since $\sigma_0(x) = \sigma(\widehat{x})$ for every $x \in \mathcal{A}(a)$, and since the spectrum of an element of $C(X)$ coincides with its range (as a function on X), it follows from (1.7) and (1.8) that

$$\sigma(a_{0,\omega}) \subseteq \sigma_0(a_{0,\omega}) = \sigma(\widehat{a_{0,\omega}}) \subset \{z \in \mathbb{C}\colon\ z = re^{i\alpha},\ r \geq 0,\ |\alpha| \leq \omega\pi\}.$$

Observe that $\sigma(a_{0,\omega})$ cannot contain points on the open rays $\{z \in \mathbb{C}\colon\ z = re^{\pm i\omega\pi},\ r > 0\}$, because this would contradict the hypothesis (1.1), in view of the spectral mapping theorem.

The uniqueness statement follows from the functional calculus: If $\omega = \frac{1}{m}$, where m is a positive integer, and if $c_\omega \in \mathcal{A}$ is also a ω-th power of a with the property

$$\sigma(c_\omega) \subseteq \{z \in \mathbb{C}\colon\ z = re^{i\alpha},\ r > 0,\ |\alpha| < \omega\pi\},$$

then

$$c_\omega = h(c_\omega^m) = h(a) = b_\omega, \qquad h(z) = z^\omega,$$

where in the first equality we have used the property that the functional calculus respects composition of functions (see, e.g., Section VII.3 in [5]). ☐

In general, b_ω in Theorem 1.2 is not unique: the $n \times n$ zero matrix ($n \geq 2$) has a continuum of p-th roots, for $p = 2, 3, \ldots$.

Corollary 1.3 *Denote by $Q(\alpha)$, $0 < \alpha < \pi$, the set of all elements $a \in \mathcal{A}$ such that the set $V(a)$ is contained in the wedge*

$$\{z = re^{i\theta} \in \mathbb{C}\colon\ r \geq 0,\ -\alpha \leq \theta \leq \alpha\}.$$

Then for every fixed ω, $0 < \omega < 1$, there exists a constant $K > 0$ such that

$$\|b_\omega(a') - b_\omega(a'')\| \leq K\|a' - a''\|^\omega$$

for every $a', a'' \in Q(\alpha)$.

The proof may be obtained as a by-product of the proof of Theorem 1.2. Note that [15] gives (in a slightly different set-up) a numerical value of the constant K.

2 Hermitian Banach *-algebras setting

Theorem 1.2 does not provide information about the numerical range of b_ω. The right setting for such results is in the *Hermitian Banach *-algebras*, with the numerical range changed to the *-numerical range. We recall the basic definitions; [17], [18] are compehensive reference works on this subject. A Banach algebra $\mathcal{A}$ is called *Banach *-algebra* if a conjugate linear involution $^* : \mathcal{A} \to \mathcal{A}$ is introduced in $\mathcal{A}$ such that $(xy)^* = y^*x^*$ for all $x, y \in \mathcal{A}$. Then $e^* = e$. If $\mathcal{A}$ is a Banach *-algebra, an element $x \in \mathcal{A}$ is called *Hermitian* if $x = x^*$. A Banach *-algebra is called *Hermitian Banach *-algebra* if every Hermitian element has real spectrum. In the rest of this section, $\mathcal{A}$ will stand for a fixed Hermitian Banach *-algebra.

The standard functional calculus leads to the following well–known statement.

Lemma 2.1 *A Hermitian element with positive spectrum admits a Hermitian square root.*

Denote by S_* the set of bounded linear functionals f on $\mathcal{A}$ such that $f(e) = \|f\| = 1$ and $f(xx^*) \geq 0$ for every $x \in \mathcal{A}$.

The *-numerical range of $a \in \mathcal{A}$ is defined as follows:

$$V_*(a) = \{f(a) \,:\, f \in S_*\} \subseteq \mathbb{C}.$$

Clearly, $V_*(a) \subseteq V(a)$. If $\mathcal{A}$ is a C^*-algebra, then in fact $V_*(a) = V(a)$. For any Banach *-algebra, the set $V_*(a)$ is compact and convex. It is easy to verify that $f(x^*) = \overline{f(x)}$ for every $f \in S_*$ and every $x \in \mathcal{A}$. Therefore, for any Hermitian $b \in \mathcal{A}$, $V_*(b) \subset \mathbb{R}$. For the converse statement to hold, that is, for all elements $x \in \mathcal{A}$ with real *-numerical range to be Hermitian, it is necessary and sufficient that

$$V_*(x) = \{0\},\ x = x^* \implies x = 0.$$

The latter property holds if and only if the involution * is *essential* (see [7]), and is of course valid for C^*-algebras, as well as in many other instances. However, it is not required for our considerations.

An element $x \in \mathcal{A}$ is called *uniformly positive* if there exists $\varepsilon > 0$ such that $z \geq \varepsilon$ for every $z \in V_*(x)$. The set of uniformly positive elements is a convex cone.

Proposition 2.2 *If $b \in \mathcal{A}$ is invertible, then b^*b is uniformly positive.*

Proof. Choose a positive $\delta < \|(bb^*)^{-1}\|$. Then the element $e - \delta(bb^*)^{-1}$ is Hermitian. Its spectrum lies in the 1-neighborhood of 1 and, being real, is therefore positive. Due to Lemma 2.1, there exists a Hermitian square root x of $e - \delta(bb^*)^{-1}$. Consequently,

$$b^*b - \delta e = b^*(e - \delta(bb^*)^{-1})b = b^*x^2b = (xb)^*(xb).$$

From the definition of S_*, then

$$0 \leq f(b^*b - \delta e) = f(b^*b) - \delta \text{ for any } f \in S_*.$$

In other words, $V_*(b^*b) \subset [\delta, +\infty)$. □

Elements of the form b^*b with invertible b are, of course, invertible. The following example shows that, in general, uniformly positive elements do not have to be invertible.

Example 2.3 Let $\mathcal{A}$ be the algebra of 2×2 matrices with the conjugate transpose as the involution *, and with the norm

$$\|a\| = \max\{\ell_1(ax) : \ell_1(x) \leq 1\}.$$

Identify $f \in \mathcal{A}^*$ with elements in $\mathcal{A}$ such that

$$f(a) = (a, f) := \operatorname{tr}(af^*),$$

where tr a stands for the trace of a matrix a. It is well known that the dual norm of ℓ_1 is the ℓ_∞ norm, and (see [1, Chapter 3]) the set S of states is the convex hull of the set of extreme vector states, i.e.,

$$S = \text{conv}\,\{yx^* : y \in \mathcal{E}_\infty, x \in \mathcal{E}_1, y^*x = 1\}, \tag{2.1}$$

where "conv" denotes "the convex hull of", and

$$\mathcal{E}_\infty = \{(y_1, y_2)^t \in \mathbb{C}^2 : |y_1| = |y_2| = 1\} \quad \text{and} \quad \mathcal{E}_1 = \{(x_1, x_2)^t \in \mathbb{C}^2 : |x_1| + |x_2| = 1\}.$$

Thus

$$V(a) = \text{conv}\,\{y^*ax \,: y \in \mathcal{E}_\infty, x \in \mathcal{E}_1, y^*x = 1\}.$$

Since $f \in S$ is an element of S_* if and only if

$$f(a^*a) = (a^*a, f) \geq 0 \qquad \text{for all } a \in \mathcal{A},$$

we see that S_* consists of all the positive semidefinite matrices in S, and

$$V_*(a) = \{(a, f) : f \in S_*\}$$

consists of all the numbers of the form (a, f), where f is a positive semidefinite matrix in S. Suppose

$$a = \begin{bmatrix} 4 & 2 \\ 2 & 1 \end{bmatrix}.$$

Then 5 is an eigenvalue of a and $(a, p) = 5$ for a positive semidefinite p if and only if $p = a/5$. Clearly, $a/5 \notin S$. Furthermore, since S has the form (2.1), if $f = (f_{ij}) \in S$ then $|f_{12}| \leq |f_{22}|$. So, $a/5 \notin S_*$. As a result, $(a, a/5) = 5 \notin V_*(a)$.

One can apply a similar argument to show that the other eigenvalue of a, namely, 0, is not in $V_*(a)$; alternatively, one may consider $5e - a$. So, $V_*(a)$ is a closed interval in $(0, 5)$. Thus, a delivers an example of a non-invertible uniformly positive element. Observe also that yet another familiar property fails on the element a, namely, the uniform positivity of the products u^*au, where a is uniformly positive and u is invertible. To this end, choose $\varepsilon > 0$ such that $V_*(a - \varepsilon e) \subseteq (0, 5)$. Let u be unitary such that

$$\tilde{a} = u^*(a - \varepsilon e)u = \begin{bmatrix} 5 - \varepsilon & 0 \\ 0 & -\varepsilon \end{bmatrix}.$$

Then $f = e_2 e_2^* \in S_*$ and thus, $(\tilde{a}, f) = -\varepsilon \in V_*(\tilde{a})$. Hence, $V_*(a - \varepsilon e) \subseteq (0, \infty)$ but $V_*(u^*(a - \varepsilon e)u) \not\subseteq [0, \infty)$.

This example is a manifestation of a general phenomenon described in Theorem 2.4 below. To formulate it, we need to fix some notation. Let ν be a norm on $\mathbb{C}^n$. Its dual norm ν^D on $\mathbb{C}^n$ is defined by

$$\nu^D(y) = \max\{|x^*y| \colon x \in \mathbb{C}^n,\ \nu(x) = 1\},$$

and the norm $\|\cdot\|_\nu$ on M_n, the algebra of $n\times n$ complex matrices with the conjugate transpose as the * operation, induced by ν is defined by

$$\|a\|_\nu = \max\{\nu(ax)\colon x \in \mathbb{C}^n,\ \nu(x) \le 1\}.$$

Identify every $f \in M_n$ with the linear functional $a \mapsto \mathrm{tr}\,(af^*)$ on M_n. Then the set S of states of $(M_n, \|\cdot\|_\nu)$ is the convex hull of the set of vector states

$$\mathcal{R} := \{yx^* \in M_n\colon 1 = \nu(x) = \nu^D(y) = x^*y\}$$

(see [14, Corollary 2.2], for example). The set S_* consists of matrices in S that are positive semi-definite.

Theorem 2.4 *Let ν be a norm on $\mathbb{C}^n$ not equal to a multiple of the ℓ_2 norm, and let $\|\cdot\|_\nu$ be the corresponding induced norm on M_n. Suppose a^* denotes the conjugate transpose of $a \in M_n$. Then there exists a vector $x \in \mathbb{C}^n$ such that $x^*x = \mathrm{tr}\,(xx^*) = 1$ and $xx^* \notin S_*$. Consequently, $b = e - xx^* \in M_n$ is singular, and $V_*(b) \subseteq (0,\infty)$ does not contain the spectrum of b. Moreover, if there exists a singular matrix in S_*, then there exists a unitary u such that $0 \in V_*(ubu^*)$.*

Observe that there are many norms ν on $\mathbb{C}^n$ such that S_* contains singular matrices. For example, if ν is a symmetric norm on $\mathbb{C}^n$, then S_* always contains E_{11}.

Proof. First, note that $\mathrm{tr}\,(xy^*) \le \nu(y)\nu^D(x)$ for every $x, y \in \mathbb{C}^n$. Suppose S_* contains xx^* for any vector $x \in \mathbb{C}^n$ with $\mathrm{tr}\,(xx^*) = 1$. Then for any $x \in \mathbb{C}^n$ with $\mathrm{tr}\,(xx^*) = 1$, xx^* can be written as a convex combination of matrices in $\mathcal{R}$. Thus, there exist positive numbers $t_1, \ldots, t_k$ summing up to one such that either $xx^* = x(\sum_{j=1}^k t_j v_j)^*$ with $\nu(v_j)\nu^D(x) = 1$, or $xx^* = (\sum_{j=1}^k t_j u_j)x^*$ with $\nu^D(u_j)\nu(x) = 1$. In the former case, we have $x = \sum_{j=1}^k t_j v_j$, and hence

$$1 = \mathrm{tr}\,(xx^*) \le \sum_{j=1}^k t_j \mathrm{tr}\,(xv_j^*) \le \sum_{j=1}^k t_j \nu^D(x)\nu(v_j) = 1,$$

and

$$\mathrm{tr}\,(xx^*) = \sum_{j=1}^k t_j \nu^D(x)\nu(v_j) \ge \nu^D(x)\nu(x).$$

It follows that $\nu^D(x)\nu(x) = 1$ (in the latter case, analogous arguments can be used to prove this equality) for any $x \in \mathbb{C}^n$ with $\mathrm{tr}\,(xx^*) = 1$. In other words, there is a support plane of the unit norm ball of ν in $\mathbb{C}^n$ at $x/\nu(x)$ with normal vector in the direction of x. This can only happen if ν is a multiple of the ℓ_2 norm (this fact is a particular case of a much more general result [13, Theorem 3]), which is a contradiction.

Now, suppose $x \in \mathbb{C}^n$ satisfies $\mathrm{tr}\,(xx^*) = 1$ and $xx^* \notin S_*$, and suppose $b = e - xx^*$. Then for any $f \in S_*$, which is a positive semidefinite matrix with trace one, we have $\mathrm{tr}\,(bf) = 1 - \mathrm{tr}\,(xx^*f) > 0$. Thus, $V_*(b) \subseteq (0,\infty)$.

Furthermore, if $f \in S_*$ is singular, and $y \in \mathbb{C}^n$ satisfies $y^*y = 1$ and $fy = 0$, then there exists a unitary u such that $ux = y$, so that $ubu^* = e - yy^*$. Clearly, $\operatorname{tr}(ubu^*f) = 1 - y^*fy = 0 \in V_*(ubu^*)$. □

For many Hermitian Banach *-algebras, however, the situation of Theorem 2.4 does not occur, that is, all uniformly positive elements there automatically are invertible. This is true, for instance, for all C^*-algebras. The Wiener algebra W of all continuous on the unit circle functions with absolutely convergent Fourier series and the norm $\|\sum c_j e^{ikx}\| = \sum |c_j|$ also has this property (due to Wiener's theorem, see [7]), though it is not a C^*-algebra. Its continuous analogue – the algebra APW of all almost periodic Bohr functions with absolutely convergent Bohr-Fourier series, – delivers yet another example of this kind, see [3]. From now on, we impose the invertibility of uniformly positive Hermitian elements as an additional requirement on the algebra $\mathcal{A}$ under consideration.

Hypothesis 2.5 *If $a = a^*$ and $V_*(a) \subset (0, +\infty)$, then a is invertible.*

One can think of Hypothesis 2.5 as a weaker version of the spectral inclusion property. As the following proposition shows, it in fact implies the spectral inclusion property for V_* in its full strength.

Proposition 2.6 *Assume the Hypothesis 2.5 is satisfied. Then, for any $a \in \mathcal{A}$, $\sigma(a) \subset V_*(a)$.*

Proof. Let us show first that a uniformly positive Hermitian element a has a positive spectrum. The spectrum $\sigma(a)$ is a priori real, and does not contain zero due to Hypothesis 2.5. For any $\lambda < 0$, $V_*(a - \lambda e) = V_*(a) - \lambda \subset (0, +\infty)$. Thus, $a - \lambda e$ is uniformly positive together with a itself, and is therefore invertible.

We now turn to the general case. It suffices to show that all elements $a \in \mathcal{A}$ with $0 \notin V_*(a)$ are invertible. Multiplying a by an appropriate non-zero scalar and using convexity of $V_*(a)$, we may without loss of generality suppose that $V_*(a)$ is contained in the open right half plane $\mathbb{C}_+$. But then $a = b + ic$ where b is uniformly positive, and both b and c are Hermitian. As was shown earlier, the spectrum of b is positive. According to Lemma 2.1, b admits a Hermitian square root x. Then

$$a = x^2 + ic = x(e + ix^{-1}cx^{-1})x = ix(x^{-1}cx^{-1} - ie)x.$$

Since the element $x^{-1}cx^{-1}$ is Hermitian together with x and c, its spectrum is real. Thus, $x^{-1}cx^{-1} - ie$ is invertible, and so is a.

□

Proposition 2.7 *Assume Hypothesis 2.5 is satisfied. Let $a \in \mathcal{A}$ be invertible and such that $V_*(a)$ is contained in the closed right halfplane. Then $V_*(a^{-1})$ is also contained in the closed right halfplane.*

Proof. Write $a = b + ic$, where b and c are Hermitian. Since $f(b) = \Re f(a)$ for all $f \in S_*$, the $*$-numerical range of b is non-negative. Thus, $b + \varepsilon e$ is uniformly positive, for every $\varepsilon > 0$. Now

$$(a + \varepsilon e)(a + \varepsilon e)^{-1}(a + \varepsilon e)^* = (a + \varepsilon e)^* = (b + \varepsilon e) - ic,$$

and (for $\varepsilon > 0$ sufficiently close to zero)

$$(a + \varepsilon e)^{-1} = (a + \varepsilon e)^{-1}(b + \varepsilon e)\left((a + \varepsilon e)^{-1}\right)^* - i(a + \varepsilon e)^{-1}c\left((a + \varepsilon e)^{-1}\right)^*.$$

Due to Proposition 2.6, the spectrum of the uniformly positive element $b + \varepsilon e$ is positive. Let x be its Hermitian square root which exists due to Lemma 2.1. Then

$$(a + \varepsilon e)^{-1}(b + \varepsilon e)\left((a + \varepsilon e)^{-1}\right)^* = zz^*, \text{ where } z = (a + \varepsilon e)^{-1}x,$$

so that its $*$-numerical range is non-negative. Hence, $V_*((a+\varepsilon e)^{-1})$ is contained in the closed right halfplane. Passing to the limit when $\varepsilon \to 0$, we obtain the required property. □

We are now ready to establish the $*$-numerical range behavior of the fractional powers considered in Section 1.

Theorem 2.8 *Let $\mathcal{A}$ be a Banach *-algebra satisfying Hypothesis* 2.5, *and let $a \in \mathcal{A}$ be such that* (1.1) *holds. Then for every $\omega \in (0, 1)$ there exists b_ω – the ωth power of a – such that $V_*(b_\omega)$ lies in the sector*

$$S_\omega = \{re^{i\theta} : r \geq 0, \ |\theta| \leq \omega\pi\}.$$

In fact, $V_*(b_\omega)$ even lies inside a certain sector with the opening $\omega\pi$. For ω being a reciprocal of an integer the element b_ω satisfying the containment condition $V_*(b_\omega) \subset S_\omega$ is unique.

Proof. *Existence.* It suffices to show that for elements $a \in \mathcal{A}$ with

$$V(a) \subset \{z : \text{Im} z \geq 0\} \tag{2.2}$$

there exists the ωth power of a, say b_ω, such that

$$V_*(b_\omega) \subset \{re^{i\theta} : r \geq 0, \ 0 \leq \theta \leq \omega\pi\}. \tag{2.3}$$

Indeed, for any a satisfying (1.1) it would then be possible to use the representation $a = a_0 e^{i\alpha}$ with $V(a_0)$ lying in the upper half plane and $-\pi \leq \alpha \leq 0$, and then choose the ωth power of a as the product of the ωth power of a_0 by $e^{i\alpha\omega}$.

So, without loss of generality we may suppose (2.2). Temporarily, let us impose a stronger condition that $V(a)$ lies in the *open* upper half plane; this restriction will be removed later. Under this condition a is of course invertible, and the standard ωth power of a, obtained with the use of functional calculus, can be represented as

$$b_\omega = -\frac{1}{2\pi i}\int_{\Gamma_{R,r}} \lambda^\omega\left((a - \lambda e)^{-1} + \lambda^{-1}e\right)\,d\lambda. \tag{2.4}$$

Here $\Gamma_{R,r}$ is the counterclockwise oriented contour consisting of the half circles $Re^{i\theta}$, $re^{i\theta}$ $(0 \le \theta \le \pi)$ and line segments $[r, R]$, $[-R, -r]$ with such a choice of $(0 <) r < R$ that $\sigma(a)$ lies inside $\Gamma_{R,r}$. (Of course, the summand $\lambda^{-1}e$ does not change the value of the integral (2.4) but it is used to improve the convergence when later we let $R \to \infty$.)

Observe that the mirror image $-\Gamma_{R,r}$ of the curve $\Gamma_{R,r}$ does not contain any singularities of $(a - \lambda e)^{-1} + \lambda^{-1}e$ in its interior. Thus,

$$0 = -\frac{1}{2\pi i} \int_{-\Gamma_{R,r}} \lambda^{\omega} \left((a - \lambda e)^{-1} + \lambda^{-1}e\right) d\lambda. \tag{2.5}$$

Multiplying (2.4) and (2.5) by $e^{i\xi}$ and $e^{-i\xi}$, respectively (at the moment, ξ is arbitrary; certain conditions on its choice will be imposed shortly), adding, and taking the limit in the right hand side when $r \to 0$, $R \to \infty$ (note that the integrals along the half circles then tend to zero):

$$\begin{aligned} e^{i\xi} b_{\omega} = \frac{1}{2\pi i}(e^{-i\xi} - e^{i\xi}) \int_0^{\infty} x^{\omega} \left((a - xe)^{-1} + x^{-1}e\right) dx \\ + \frac{1}{2\pi i}\left(e^{-i(\xi+\omega\pi)} - e^{i(\xi+\omega\pi)}\right) \int_{-\infty}^{0} |x|^{\omega} \left((a - xe)^{-1} + x^{-1}e\right) dx = \\ -\frac{\sin\xi}{\pi} \int_0^{\infty} x^{\omega} \left((a - xe)^{-1} + x^{-1}e\right) dx - \frac{\sin(\xi + \omega\pi)}{\pi} \int_{-\infty}^{0} |x|^{\omega} \left((a - xe)^{-1} + x^{-1}e\right) dx. \end{aligned}$$

Therefore, for any $f \in S_*$:

$$\begin{aligned} \operatorname{Im} f(e^{i\xi} b_{\omega}) = -\frac{\sin\xi}{\pi} \int_0^{\infty} x^{\omega} \operatorname{Im} f\left((a - xe)^{-1}\right) dx \\ - \frac{\sin(\xi + \omega\pi)}{\pi} \int_{-\infty}^{0} |x|^{\omega} \operatorname{Im} f\left((a - xe)^{-1}\right) dx. \end{aligned} \tag{2.6}$$

Due to (2.2), $V_*(a - xe)$ $(= V_*(a) - x)$ lies in the upper half plane for any $x \in \mathbb{R}$. Applying Proposition 2.7 to $-i(a - xe)$, we conclude that $V_*\left((a - xe)^{-1}\right)$ lies in the *lower* half plane. Thus, for all $\xi \in [0, (1 - \omega)\pi]$ formula (2.6) implies that $\operatorname{Im} f(e^{i\xi} b_{\omega}) \ge 0$. In other words, (2.3) holds.

Consider now an arbitrary element $a \in \mathcal{A}$ satisfying (2.2). Think of it as a limit of the elements $a_{\varepsilon} = a + i\varepsilon e$ when $\varepsilon \downarrow 0$. As we just found out, for each of a_{ε} the ω-th power constructed as in Theorem 1.2 has $*$-numerical range satisfying (2.3). Using the continuity of the ω-th power of x as a function of x (Corollary 1.3 applied to $-ia$) and the continuity of $V_*(z)$ as a function of z, we see that the same inclusion (2.3) holds after taking the limit.

Uniqueness for $\omega = 1/m$, m positive integer. In case of invertible $a \in \mathcal{A}$, it follows from Theorem 1.2. Suppose now that for a (naturally, non-invertible) element $a \in \mathcal{A}$ satisfying (1.1) there exist $c_1, c_2 \in \mathcal{A}$ such that $V_*(c_j) \subset S_{\omega}$, $c_j^m = a$ $(j = 1, 2)$. Let $b_{\omega}(\epsilon, j) = c_j + \epsilon e$.

Then $b_\omega(\epsilon, j)$ is the ω-th power of the (invertible) element $(c_j + \epsilon e)^m$ with $V_*(b_\omega(\epsilon, j)) \subset S_\omega$. If ϵ is small enough, then $(c_j + \epsilon e)^m$ satisfies (1.1) together with a, so that $b_\omega(\epsilon, j)$ must satisfy the inequality (Corollary 1.3):

$$\|b_\omega(\epsilon, 1) - b_\omega(\epsilon, 2)\| \leq K\|(c_1 + \epsilon e)^m - (c_2 + \epsilon e)^m\|^\omega.$$

Letting $\epsilon \downarrow 0$ we see that the right hand side of the latter inequality converges to 0 while the left hand side converges to $\|c_1 - c_2\|$. Thus, $c_1 = c_2$. □

For the case of square roots, that is, $\omega = 1/2$, a different approach to the proof of Theorem 2.8 is possible. It is based on the Lyapunov's theorem on the uniform positivity of the (unique) solution $w \in \mathcal{A}$ of the equation

$$wa + a^* w = h$$

for $a \in \mathcal{A}$ with the spectrum in $\mathbb{C}_+$, and in the matrix case was utilized in [9]. A treatment of Lyapunov's theorem in the Hermitian Banach *-algebra setting can be found in [19].

References

[1] F. F. Bonsall and J. Duncan, *Numerical ranges of operators on normed spaces and of elements of normed algebras*, Cambridge University Press, London, 1971.

[2] ———, *Numerical ranges. II*, Cambridge University Press, New York, 1973, London Mathematical Society Lecture Notes Series, No. 10.

[3] A. Böttcher, Yu. I. Karlovich, and I. M. Spitkovsky, *Convolution operators and factorization of almost periodic matrix functions*, Birkhäuser Verlag, Basel and Boston, 2002.

[4] J. B. Conway, *A course in functional analysis*, second ed., Springer-Verlag, New York, 1990.

[5] N. Dunford and J. T. Schwartz, *Linear operators. Part I*, John Wiley & Sons Inc., New York, 1988, General theory, With the assistance of William G. Bade and Robert G. Bartle, Reprint of the 1958 original, A Wiley-Interscience Publication.

[6] G. M. Fikhtengol'ts. *The fundamentals of mathematical analysis*, Vol. I, II. Pergamon Press, 1965 (Translation from Russian).

[7] I. Gelfand, D. Raikov, and G. Shilov, *Commutative normed rings*, Chelsea Publishing Co., New York, 1964.

[8] C. R. Johnson and M. Neumann, *Square roots with positive definite Hermitian part*, Linear and Multilinear Algebra **8** (1979/80), no. 4, 353–355.

[9] C. R. Johnson and K. Okubo, *Uniqueness of matrix square roots under a numerical range condition*, Linear Algebra Appl. **341** (2002), 195–199.

[10] C. R. Johnson, K. Okubo, and R. Reams, *Uniqueness of matrix square roots and an application*, Linear Algebra Appl. **323** (2001), no. 1-3, 51–60.

[11] T. Kato, *Perturbation theory for linear operators*, Springer-Verlag, Berlin, 1995, Reprint of the 1980 edition.

[12] H. Langer, *Über die Wurzeln eines maximalen dissipativen operators*, Acta Math. Acad. Sci. Hungar. **13** (1962), 415–424.

[13] C. K. Li, E. Poon, and H. Schneider, *Induced norms, states, and numerical ranges*, submitted for publication.

[14] C. K. Li and A. R. Sourour, *Linear operators on matrix algebras that preserve the numerical range, numerical radius, or the states*, Canadian J. of Math., to appear.

[15] V. I. Macaev and Ju. A. Palant, *On the powers of a bounded dissipative operator*, Ukrain. Mat. Zh. **14** (1962), 329–337. (Russian).

[16] D. W. Masser and M. Neumann, *On the square roots of strictly quasi-accretive complex matrices*, Linear Algebra Appl. **28** (1979), 135–140.

[17] T. W. Palmer, *Banach algebras and the general theory of* **-algebras. Vol.I: Algebras and Banach algebras*, Cambridge University Press, 1994.

[18] T. W. Palmer, *Banach algebras and the general theory of* **-algebras. Vol.II*, Cambridge University Press, 2001.

[19] M. Sonis, *Localization of left and right spectra in Banach algebras*, Funct. Differential Equations Israel Sem. **2** (1994), 195–211 (1995).

[20] R. R. Smith, *On Banach algebra elements of thin numerical range*, Math. Proc. Cambridge Philos. Soc. **86** (1979), 71–83.

[21] J. G. Stampfli and J. P. Williams, *Growth conditions and the numerical range in a Banach algebra*, Tôhoku Math. J. (2) **20** (1968), 417–424.