Inequalities and Equalities for the Cartesian Decomposition of Complex Matrices[1]

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Abstract

Let A and B be Hermitian matrices and let $C = A + iB$. Inequalities and equalities for the eigenvalues, singular values of the matrices A, B, and C are discussed. Known results on inequalities are surveyed, new results on equality cases are proved, and open problems are mentioned.

1 Introduction

Let M_n denote the set of all $n \times n$ complex matrices. For any $C \in M_n$, we can write $C = A + iB$, in which $A = (C + C^*)/2$ and $B = i(C^* - C)/2$ are both Hermitian. This is called the *Cartesian decomposition* of C. We discuss inequalities and equalities involving the eigenvalues and singular values of A, B, and C. We survey known results on inequalities, prove new results on equality cases, and mention some open problems.

Given an $X \in M_n$, let $s(X) = (s_1(X), \ldots, s_n(X))$ be the vector of singular values of X with $s_1(X) \geq \cdots \geq s_n(X)$, and let $\lambda(X) = (\lambda_1(X), \ldots, \lambda_n(X))$ be a vector of eigenvalues of X. If X is Hermitian, we assume that $\lambda_1(X) \geq \cdots \geq \lambda_n(X)$.

General references on matrix inequalities are [1, 11, 12, 19]. Some equality cases of matrix inequalities have been studied in [5, 13, 14, 16]. We use the following notation for majorization in our discussion [19]. For two real vectors x and y in $\mathbf{R}^n$, if the sum of the m largest entries of x is not larger than that of y for each $m = 1, \ldots, n$, we write

$$x \prec_w y; \tag{1.1}$$

if, in addition, the sum of all the entries of x is the same as that of y, we write

$$x \prec y. \tag{1.2}$$

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The relation (1.1) is called *weak majorization*; the relation (1.2) is called *majorization.* Denote by $x \circ y$ the entrywise (Hadamard) product of two vectors. Let $x_\downarrow$ and $x_\uparrow$ denote the vectors obtained from the real vector x by rearranging its entries in descending and ascending order, respectively.

For a complex vector $z = (z_1, \ldots, z_n)$ we write $|z| = (|z_1|, \ldots, |z_n|)$, $\operatorname{Re} z = (z + \bar{z})/2$, and $\operatorname{Im} z = i(\bar{z} - z)/2$.

Many of our results are valid for compact operators acting on separable Hilbert spaces. Also, if the results do not involve complex numbers, they are often valid for real matrices or operators as well.

The following results, which are of independent interest, are used frequently in our study.

Proposition 1.1 *Let A, B, and $C = A + B$ be $n \times n$ complex matrices, and let $s(A) = (a_1, \ldots, a_n)$, $s(B) = (b_1, \ldots, b_n)$, and $s(C) = (c_1, \ldots, c_n)$. For every integer $k \in \{1, \ldots, n\}$,*

$$\sum_{j=1}^{k} c_j \leq \sum_{j=1}^{k} (a_j + b_j). \tag{1.3}$$

Equality holds in (1.3) *for some k if and only if there exist unitary matrices U and V such that $UAV = A_1 \oplus A_2$ and $UBV = B_1 \oplus B_2$, where A_1 and B_1 are positive semi-definite matrices with eigenvalues $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$, respectively.*

Proof. By the singular value decomposition [11, p. 414], we may assume that $C = \operatorname{diag}(c_1, \ldots, c_k) \oplus C_2$. Let $A = (a_{ij})$ and $B = (b_{ij})$. By [23, Theorem 1], for every integer $k \in \{1, \ldots, n\}$,

$$\sum_{j=1}^{k} c_j = \sum_{j=1}^{k} (a_{jj} + b_{jj}) = \sum_{j=1}^{k} a_{jj} + \sum_{j=1}^{k} b_{jj} \leq \sum_{j=1}^{k} a_j + \sum_{j=1}^{k} b_j. \tag{1.4}$$

Equality holds in (1.4) if and only if $\sum_{j=1}^{k} a_{jj} = \sum_{j=1}^{k} a_j$ and $\sum_{j=1}^{k} b_{jj} = \sum_{j=1}^{k} b_j$. The result follows from [16, Corollary 3.2]. □

The inequality (1.3) is the triangle inequality for the Ky Fan k-norms [11, Section 3.4]. The key to the rest of the proof of Proposition 1.1 is the case of equality in Von Neumann's celebrated trace inequality [12, Section 3.1, Problem 4].

Proposition 1.2 *Let A, B, and $C = A + B$ be $n \times n$ Hermitian matrices with $s(A) = (a_1, \ldots, a_n)$, $s(B) = (b_1, \ldots, b_n)$, and $s(C) = (c_1, \ldots, c_n)$. For every integer $k \in \{1, \ldots, n\}$,*

$$\sum_{j=1}^{k} c_j \leq \sum_{j=1}^{k} (a_j + b_j). \tag{1.5}$$

Equality holds in (1.5) *for some k if and only if there exists a unitary matrix U such that $U^*AU = A_1 \oplus A_2 \oplus A_3$ and $U^*BU = B_1 \oplus B_2 \oplus B_3$, where A_1 and B_1 are positive semi-definite matrices of the same size, A_2 and B_2 are negative semi-definite matrices of the same size, and the $k \times k$ matrices $A_1 \oplus A_2$ and $B_1 \oplus B_2$ have singular values $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$, respectively.*

Proof. The inequality in (1.5) follows from Proposition 1.1. We need to consider only the case of equality. ($\Leftarrow$) By direct verification.

($\Rightarrow$) As C is Hermitian, its singular values are the absolute values of its eigenvalues. Let U be a unitary matrix U such that $U^*CU = \operatorname{diag}(\gamma_1, \ldots, \gamma_n)$ with $\gamma_1 \geq \cdots \geq \gamma_r > 0 \geq \gamma_{r+1} \geq \cdots \geq \gamma_k$ satisfying $\{|\gamma_1|, \ldots, |\gamma_k|\} = \{s_1(C), \ldots, s_k(C)\}$. Then for $D = I_r \oplus -I_{k-r} \oplus I_{n-k}$ we have $DU^*CU = \operatorname{diag}(c_{j_1}, \ldots, c_{j_k}) \oplus C_2$, where $(j_1, \ldots, j_k)$ is a permutation of $(1, \ldots, k)$. Proposition 1.1 shows that $DU^*AU = A_1 \oplus A_2$ and $DU^*BU = B_1 \oplus B_2$, where A_1 and B_1 are positive semi-definite matrices with eigenvalues $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$, respectively. Since $D(A_1 \oplus A_2) = U^*AU$ is Hermitian, $(I_r \oplus -I_{k-r})A_1$ is Hermitian. This means that $I_r \oplus -I_{k-r}$ commutes with A_1; so A_1 must be in block diagonal form: $A_1 = A' \oplus A''$, where A' is an $r \times r$ positive semi-definite matrix and A'' is a $(k-r) \times (k-r)$ negative semi-definite matrix. A similar argument shows that B_1 is also of the same form. $\square$

2 Eigenvalues

In this section, we survey some results and problems involving the eigenvalues of the Hermitian matrices A and B, and those of the matrix $C = A + iB$. The majorization relations in the following theorem were proved in [7, 2], and the equality cases were treated in [16].

Theorem 2.1 *Suppose $x, y \in \mathbf{R}^n$ and $z \in \mathbf{C}^n$.*

(a) *There exists a $C \in M_n$ such that $\lambda(C) = z$ and $\lambda(C + C^*) = 2x$ if and only if $\operatorname{Re} z \prec x$.*

(b) *There exists a $C \in M_n$ such that $\lambda(C) = z$ and $i\lambda(C^* - C) = 2y$ if and only if $\operatorname{Im} z \prec y$.*

Furthermore, suppose $A, B \in M_n$ are Hermitian, $C = A + iB$, and $1 \leq k < n$.

(i) *$\sum_{j=1}^k \operatorname{Re} \lambda_j(C) = \sum_{j=1}^k \lambda_j(A)$ if and only if C is unitarily similar to $C_1 \oplus C_2$, where $C_1 \in M_k$ satisfies $\lambda(C_1) = (\lambda_1(C), \ldots, \lambda_k(C))$ and $\lambda(C_1 + C_1^*) = 2(\lambda_1(A), \ldots, \lambda_k(A))$.*

(ii) *$\sum_{j=1}^k \operatorname{Im} \lambda_j(C) = \sum_{j=1}^k \lambda_j(A)$ if and only if C is unitarily similar to $C_1 \oplus C_2$, where $C_1 \in M_k$ satisfies $\lambda(C_1) = (\lambda_1(C), \ldots, \lambda_k(C))$ and $i\lambda(C_1^* - C_1) = 2(\lambda_1(B), \ldots, \lambda_k(B))$.*

Problem 2.2 *Determine necessary and sufficient conditions on $x, y \in \mathbf{R}^n$ and $z \in \mathbf{C}^n$ for the existence of a $C = A + iB \in M_n$ with $\lambda(A) = x$, $\lambda(B) = y$, and $\lambda(C) = z$.*

Clearly, the conditions

$$\operatorname{Re} z \prec x \quad \textit{and} \quad \operatorname{Im} z \prec y \tag{2.1}$$

are necessary, but they are not sufficient even for 2×2 matrices.

Example 2.3 Take $x = (1, 1)$, $y = (2, 0)$, and $z = (1 + i, 1 + i)$. If $C = A + iB \in M_2$ has $\lambda(A) = (1, 1)$ and $\lambda(B) = (2, 0)$, then $A = I_2$, so C is normal with eigenvalues $1 + 2i$ and 1.

Here is an additional necessary condition obtained in [15] (see also [20] and [10]).

Theorem 2.4 *Let $A, B \in M_n$ be Hermitian and let $C = A+iB$. Write $\lambda(A) = x$, $\lambda(B) = y$, and $\lambda(C) = z$. Then*

$$\mathrm{Re}\,(z_1^2, \ldots, z_n^2) \prec (x \circ x)_\downarrow - (y \circ y)_\uparrow, \tag{2.2}$$

i.e.,

$$\Big((\mathrm{Re}\, z_1)^2 - (\mathrm{Im}\, z_1)^2, \ldots, (\mathrm{Re}\, z_n)^2 - (\mathrm{Im}\, z_n)^2\Big) \prec \Big(\lambda_1(A^2) - \lambda_n(B^2), \ldots, \lambda_n(A^2) - \lambda_1(B^2)\Big).$$

For $n = 2$, (2.1) and (2.2) are necessary and sufficient conditions for the existence of $C = A + iB$ with $\lambda(A) = x$, $\lambda(B) = y$, and $\lambda(C) = z$. However, they are not sufficient if $n \geq 3$; see [15].

Example 2.5 Let $x = (100, 4, 0)$, $y = (4, 0, 0)$, and $z = (100 + 3i, 3 + i, 1)$. Then x, y, z satisfy (2.1) and (2.2). Suppose A, B, and $C = A + iB \in M_3$ are such that A and B are Hermitian, $\lambda(A) = x$, $\lambda(B) = y$, and $\lambda(C) = z$. Then there exists a unitary U such that U^*CU is in upper triangular form with diagonal entries $100 + 3i, 3 + i, 1$. Then the $(1, 1)$ entry of U^*AU is 100, which is its largest eigenvalue. So, $U^*AU = [100] \oplus A_2$. Since U^*CU is upper triangular and $100 + 3i$ is its $(1, 1)$ entry, it follows that 3 is the $(1, 1)$ entry of U^*BU and is the only nonzero entry in the first column, which is impossible.

To date, Problem 2.2 is still open for $n \geq 3$.

3 Singular Values

In this section, we focus on relations between the singular values of Hermitian matrices A and B, and those of $C = A + iB$. For general $X, Y \in M_n$ and $Z = X + Y$, there are index sets $P, Q, R \subset \{1, \ldots, n\}$ of the same size such that

$$\sum_{r\in R} s_r(Z) \leq \sum_{p\in P} s_p(X) + \sum_{q\in Q} s_q(Y).$$

One can describe a collection of such index sets in terms of Schubert calculus (or Littlewood-Richardson rules in combining Young's diagrams) so that these inequalities completely determine the relations among the singular values of matrices X, Y, Z such that $Z = X + Y$; see the survey [9] on this and several related topics, and see [3] for an exposition of these ideas at a more elementary level. For simplicity, we focus on some basic inequalities that are used frequently in applications such as perturbation theory and the theory of norms. In most of these applications, it suffices to consider the following standard inequalities of Thompson [24, Theorem 3]:

Whenever $1 \leq i_1 < \cdots < i_m \leq n$ and $1 \leq j_1 < \cdots < j_m \leq n$ are such that $i_m + j_m - m \leq n$, we have

$$\sum_{r=1}^{m} s_{i_r+j_r-r}(Z) \leq \sum_{r=1}^{m} s_{i_r}(X) + \sum_{r=1}^{m} s_{j_r}(Y).$$

We apply some of these general results to our special case $C = A + iB$, and analyze the equality cases. For notation simplicity, throughout this section we assume that

$$s(C) = (c_1, \ldots, c_n),\ s(A) = (a_1, \ldots, a_n) \text{ and } s(B) = (b_1, \ldots, b_n)$$

as in Propositions 1.1 and 1.2. The majorization relations in the following theorem were proved in [7]. We study the equality cases.

Theorem 3.1 *Suppose $A, B \in M_n$ are Hermitian and $C = A + iB$. Then*

$$(a_1, \ldots, a_n) \prec_w (c_1, \ldots, c_n) \quad \text{and} \quad (b_1, \ldots, b_n) \prec_w (c_1, \ldots, c_n).$$

Moreover, for any given $k \in \{1, \ldots, n\}$,

(a) $\sum_{j=1}^k a_j = \sum_{j=1}^k c_j$ *if and only if C is unitarily similar to $DP \oplus C_0$, where $D \in M_k$ is a diagonal orthogonal matrix, $P \in M_k$ is positive semi-definite, and* $\operatorname{tr} P = \sum_{j=1}^k c_j$*;*

(b) $\sum_{j=1}^k b_j = \sum_{j=1}^k c_j$ *if and only if C is unitarily similar to $iDP \oplus C_0$, where $D \in M_k$ is a diagonal orthogonal matrix, $P \in M_k$ is positive semi-definite, and* $\operatorname{tr} P = \sum_{j=1}^k c_j$*.*

Proof. (a) Suppose $\sum_{j=1}^k a_j = \sum_{j=1}^k c_j$. Let U be a unitary matrix such that

$$U^*AU = \operatorname{diag}(d_1, \ldots, d_k) \oplus A_2$$

with $|d_j| = a_j$ for $j = 1, \ldots, k$. If $U^*CU = (c_{ij})$ then

$$\sum_{j=1}^k c_j = \sum_{j=1}^k a_j \le \sum_{j=1}^k |c_{jj}| \le \sum_{j=1}^k c_j.$$

Thus, $c_{jj} = d_j$ for all $j = 1, \ldots, k$. By Theorem 3.1 in [16], $C = C_1 \oplus C_0$ with $C_1 \in M_k$ and there exists a diagonal orthogonal matrix (signature matrix) $D \in M_k$ such that $DC_1 = P$ is positive semi-definite with eigenvalues $c_1, \ldots, c_k$. Hence $C_1 = DP$ with $\operatorname{tr} P = \sum_{j=1}^k c_j$.

Conversely, suppose U is unitary, $U^*CU = DP \oplus C_0$, $D \in M_k$ is a diagonal orthogonal matrix, and P is positive semi-definite with $\operatorname{tr} P = \sum_{j=1}^k c_j$. If $U^*AU = (a_{ij})$ then

$$\sum_{j=1}^k c_j = \operatorname{tr} P = \sum_{j=1}^k |a_{jj}| \le \sum_{j=1}^k a_j \le \sum_{j=1}^k c_j.$$

The proof of (b) is similar. □

Theorem 3.2 *Suppose $A, B \in M_n$ are Hermitian and $C = A + iB$. For every integer $k \in \{1, \ldots, n\}$,*

$$\sum_{j=1}^k c_j \le \sum_{j=1}^k (a_j + b_j). \tag{3.1}$$

Equality holds in (3.1) *for some integer k if and only if there exists a unitary matrix U such that one of the following conditions holds:*

(a) $U^*CU = \operatorname{diag}(\alpha_1, \ldots, \alpha_s) \oplus \operatorname{diag}(i\beta_1, \ldots, i\beta_t) \oplus 0_{n-s-t}$, *where* $k \geq s+t$, *and* α_j *and* β_j *are real numbers satisfying* $|\alpha_j| = a_j$, $|\beta_j| = b_j$, $j = 1, \ldots, k$.

(b) $U^*CU = \begin{pmatrix} a_1 I_k & b_1 I_k \\ -b_1 I_k & -a_1 I_k \end{pmatrix} \oplus C'$.

Proof. ($\Leftarrow$) By direct verification.

($\Rightarrow$) We use induction on n. The result is obvious when $n = 1$. Assume that $n \geq 2$, and that the result is true for all matrices of size less than n. Suppose A and B are nonzero $n \times n$ Hermitian matrices. By Proposition 1.1, there exist unitary matrices U and V such that $U^*AV = A_1 \oplus A_2$ and $iU^*BV = B_1 \oplus B_2$, where A_1 and B_1 are positive semi-definite matrices with eigenvalues $a_1, \ldots, a_k$ and $b_1, \ldots, b_k$, respectively. We may further assume that $A_1 = \operatorname{diag}(a_1, \ldots, a_k)$. Let u_i, v_i denote the i-th columns of U and V, respectively. Since $u_1^*Av_1 = a_1$, we have 3 cases:

(i) v_1 is an eigenvector of A corresponding to the eigenvalue a_1 and $u_1 = v_1$;

(ii) v_1 is an eigenvector of A corresponding to the eigenvalue $-a_1$ and $u_1 = -v_1$;

(iii) $v_1 = e_+ + e_-$, where e_+ and e_- are eigenvectors of A corresponding to the eigenvalues a_1 and $-a_1$, respectively, and $u_1 = e_+ - e_-$.

Suppose case (i) holds. Since B is Hermitian and B_1 is positive semi-definite, we must have $u_1^*Bv_1 = 0$ and hence $B_1 = (0) \oplus B_1'$. If $k = 1$, the result follows.

Now suppose $k > 1$. Write $U^*AV = (a_1) \oplus A'$, $iU^*BV = (0) \oplus B'$, and $U^*CV = (a_1) \oplus C'$. Since $c_1 = a_1$ and $b_k = 0$, we have

$$\begin{aligned} s_1(C') + \cdots + s_{k-1}(C') &= c_1 + \cdots + c_k - a_1 \\ &= a_2 + \cdots + a_k + b_1 + \cdots b_{k-1} \\ &= s_1(A') + \cdots s_{k-1}(A') + s_1(B') + \cdots + s_{k-1}(B'). \end{aligned}$$

By the induction assumption, A' and B' satisfy one of the conditions (a) or (b). However, if A' and B' satisfy (b), we have $b_1 = s_1(B') = \cdots = s_k(B') = b_k = 0$, which is a contradiction. Thus A' and B' satisfy (a).

Next, suppose case (ii) holds. We may replace A and B by $-A$ and $-B$ and the result follows from case (i).

Finally, suppose case (iii) holds. Let E_+ and E_- denote the eigenspaces of A corresponding to eigenvalues a_1 and $-a_1$, respectively. Let $r = k$ if $a_1 = \cdots = a_k$, and let $r = s$ if $s < k$ and $a_1 = \cdots = a_s > a_{r+1}$. If $\dim E_- < r$ then, as $v_1, \ldots, v_r$ are orthonormal vectors in $E_+ \oplus E_-$, we have $\dim E_+ + \dim(\operatorname{span}\{v_1, \ldots, v_r\}) > \dim E_+ + \dim E_-$ and hence $\operatorname{span}\{v_1, \ldots, v_r\} \cap E_+ \neq \{0\}$. Thus, there exists an $r \times r$ unitary matrix W such that the first column of $V(W \oplus I)$ is in E_+. Replacing U and V by $U(W \oplus I)$ and $V(W \oplus I)$, respectively, we are back to case (i), and the result follows.

Now suppose $\dim E_- \geq r$. Using the same argument, we may also assume $\dim E_+ \geq r$. If $r < k$ then $r = \dim E_+ + \dim E_-$, which is not true. We therefore have $r = k$ and hence

$a_1 = \cdots = a_{2k}$. Replacing A and B by iA and iB, we can further assume that $b_1 = \cdots = b_{2k}$. Thus we have $c_1 + \cdots + c_k = k(a_1 + b_1)$ and hence $c_i = a_1 + b_1$, $i = 1, \ldots, k$.

Since $v_1 = e_+ + e_-$ and $u_1 = e_+ - e_-$, we have $Av_1 = a_1 u_1$ and $Au_1 = a_1 v_1$ and thus we know that $\operatorname{span}\{v_1, u_1\}$ $(= (\operatorname{span}\{e_+, e_-\})$ is an invariant subspace of A. Let $P = [e_+, e_-]$ and consider $P^*CP = P^*AP + iP^*BP$ (that is, consider the orthogonal projection of C onto $\operatorname{span}\{v_1, u_1\}$). Obviously, we have $s_1(P^*CP) = c_1$, $s_1(P^*AP) = a_1$, and $s_1(P^*BP) = b_1$. Notice that $iu_1^*Bv_1 = b_1$ implies $iv_1^*Bu_1 = -b_1$. As u_1 and v_1 are linearly independent, we deduce that $s_2(P^*BP) = b_1$. Thus, we are now dealing with the case in which $n = 2$, $k = 1$, A has eigenvalues a_1 and $-a_1$, and B has singular values $b_1 = b_2$.

Let $A = \begin{pmatrix} a_1 & 0 \\ 0 & -a_1 \end{pmatrix}$. If $B = \pm b_1 I_2$ then $c_1 = \sqrt{a_1^2 + b_1^2} < a_1 + b_1$. Thus we may assume that B has eigenvalues b_1 and $-b_1$. With a suitable unitary similarity, we may assume that $iB = \begin{pmatrix} ib & t \\ -t & -ib \end{pmatrix}$ and $t \geq 0$. Then $C = \begin{pmatrix} a_1 + ib & t \\ -t & -a_1 - ib \end{pmatrix}$. A computation reveals that $b_1 = \sqrt{b^2 + t^2}$ and $c_1 = \sqrt{a_1^2 + b^2 + t^2 + 2a_1 t}$. Consequently, $a_1 + b_1 = c_1$ if and only if $b = 0$ and $t = b_1$.

Our argument shows that we can find a unitary matrix Q whose first two columns are in $\operatorname{span}\{e_+, e_-\}$ and $Q^*AQ = \begin{pmatrix} a_1 & 0 \\ 0 & -a_1 \end{pmatrix} \oplus A'$ and $iQ^*BQ = \begin{pmatrix} 0 & b_1 \\ -b_1 & 0 \end{pmatrix} \oplus B'$. Note that $s_2\left(\begin{pmatrix} a_1 & b_1 \\ -b_1 & -a_1 \end{pmatrix}\right) < a_1 + b_1$. Thus, A' and B' satisfy

$$s_1(A' + B') + \cdots + s_{k-1}(A' + iB') = s_1(A') + \cdots + s_{k-1}(A') + s_1(B') + \cdots + s_{k-1}(B').$$

By the induction assumption, we can conclude that A' and B' satisfy one of the conditions (a) or (b). In this case, since we have $a_1 = \cdots = a_{2k}$ and $b_1 = \cdots = b_{2k}$, A' and B' satisfy (b). $\square$

In [1], it was proved that

$$(c_1, \ldots, c_n) \prec_w \sqrt{2}(|a_1 + ib_1|, ..., |a_n + ib_n|). \tag{3.2}$$

It was conjectured in [1] and was proved recently in [25] that

$$(|a_1 + ib_1|, ..., |a_n + ib_n|) \prec_w \sqrt{2}(c_1, \ldots, c_n). \tag{3.3}$$

We now study the equality cases in the following theorem.

Theorem 3.3 *Suppose $A, B \in M_n$ are Hermitian and $C = A + iB$. Then (3.2) and (3.3) hold. For any given $k \in \{1, \ldots, n\}$,*

(a) $\sum_{j=1}^k c_j = \sqrt{2} \sum_{j=1}^k |a_j + ib_j|$ *if and only if C is unitarily similar to one of the following forms:*

(a.i) $\operatorname{diag}(\gamma_1, \ldots, \gamma_r) \oplus i\operatorname{diag}(\gamma_{r+1}, \ldots, \gamma_{2r}) \oplus 0_{n-2r}$, *where* $2r \le k$ *and* $\gamma_1, \ldots, \gamma_{2r} \in \mathbf{R}$ *satisfy* $|\gamma_j| = |\gamma_{r+j}|$ *for* $j = 1, \ldots, r$,

(a.ii) $\begin{pmatrix} 0 & cI_k \\ 0 & 0 \end{pmatrix} \oplus C'$, *where* $s_1(C') \le c$;

(b) $\sqrt{2}\sum_{j=1}^k c_j = \sum_{j=1}^k |a_j + ib_j|$ *if and only if* C *is unitarily similar to one of the following forms:*

(b.i) $c_1(D_1 \oplus iD_2 \oplus C')$, *where* D_1 *and* D_2 *are* $k \times k$ *diagonal orthogonal matrices and* $s_1(C') \le 1$,

(b.ii) $\begin{pmatrix} 0 & c_1 \\ 0 & 0 \end{pmatrix} \oplus \cdots \oplus \begin{pmatrix} 0 & c_p \\ 0 & 0 \end{pmatrix} \oplus 0_{n-2p}$, *where* $2p \le k$.

Proof. (a) For any $1 \le k \le n$, we have

$$\sum_{j=1}^k c_j \le \sum_{j=1}^k a_j + \sum_{j=1}^k b_j = \sum_{j=1}^k (a_j + b_j) \le \sum_{j=1}^k \sqrt{2}|a_j + ib_j| \tag{3.4}$$

So, $\sum_{j=1}^k c_j = \sqrt{2}\sum_{j=1}^k |a_j + ib_j|$ if and only if both of the inequalities in (3.4) become equalities. The second inequality is an equality if and only if $a_i = b_i$, $i = 1, \ldots, k$. The result follows from Theorem 3.2.

(b) ($\Leftarrow$) By direct verification.

($\Rightarrow$) We divide the proof into three cases.

Case 1. $k = 1$. Assume $c_1 > 0$. We have $a_1 \le c_1$ and $b_1 \le c_1$. Hence $|a_1 + ib_1| \le \sqrt{2}c_1$ and equality holds if and only if $a_1 = b_1 = c_1$. We now suppose that $a_1 = b_1 = c_1$. Let x be a unit eigenvector of A corresponding to an eigenvalue with absolute value a_1. Let V be a unitary matrix with x as its first column. Then the $(1,1)$ entry of V^*CV is $x^*Ax + ix^*Bx$. As $|x^*Ax| = c_1$, we deduce that $x^*Bx = 0$ and furthermore that $V^*CV = (x^*Ax) \oplus C'$ and $V^*AV = (x^*Ax) \oplus A'$. Thus $V^*BV = (0) \oplus B'$ and hence $Bx = 0$. Let y be a unit eigenvector of B corresponding to an eigenvalue with absolute value b_1. As before, we have $Ay = 0$. Thus x and y are orthogonal because they are eigenvectors of A corresponding to different eigenvalues. Let U be a unitary matrix with x and y as its first two columns. Then $U^*CU = \operatorname{diag}(\pm c_1, \pm ic_1) \oplus C''$, as required.

Case 2. $k = n$. Let us consider (12) in [1]:

$$(c_1^2 + c_n^2, \ldots, c_n^2 + c_1^2)/2 \prec (a_1^2 + b_1^2, \ldots, a_n^2 + b_n^2).$$

Since $f(t) = \sqrt{t}$ is strictly concave on $[0, \infty)$, we have

$$\frac{1}{\sqrt{2}}\sum_{j=1}^n (c_j^2 + c_{n-j+1}^2)^{\frac{1}{2}} \ge \sum_{j=1}^n |a_j + ib_j|$$

and equality holds if and only if rearrangement of $(c_1^2+c_n^2,\ldots,c_n^2+c_1^2)/2$ in nonincreasing order gives $(a_1^2+b_1^2,\ldots,a_n^2+b_n^2)$. Thus, we have

$$\begin{aligned}\sum_{j=1}^n |a_j+ib_j| &\le \frac{1}{\sqrt{2}}\sum_{j=1}^n (c_j^2+c_{n-j+1}^2)^{\frac{1}{2}} &(3.5)\\ &\le \frac{1}{\sqrt{2}}\sum_{j=1}^n (c_j+c_{n-j+1}) &(3.6)\\ &= \sqrt{2}\sum_{j=1}^n c_j\end{aligned}$$

Our assumption implies that (3.5) and (3.6) are equalities. Since $\sqrt{a^2+b^2} \le |a|+|b|$ and equality holds if and only if a or b is 0, equality in (3.6) implies that $c_j = 0$ for $j=[n/2]+1,\ldots,n$. Hence, rearrangement of $(c_1^2+c_n^2,\ldots,c_n^2+c_1^2)/2$ in nonincreasing order is just $(c_1^2,c_1^2,c_2^2,c_2^2,\ldots)$, which is the same as $(a_1^2+b_1^2,\ldots,a_n^2+b_n^2)$. Thus,

$$\frac{1}{\sqrt{2}}c_j = |a_{2j-1}+ib_{2j-1}| = |a_{2j}+ib_{2j}| \quad \text{for } j=1,\ldots,[n/2],$$

and $a_n=b_n=0$ if n is odd.

Suppose $c_1>0$. Let x be a unit vector such that $||Cx||=c_1$, where $||\cdot||$ is the Euclidean norm. Since $C^*C+CC^*=2(A^2+B^2)$, we have $c_1^2+x^*CC^*x = 2(x^*A^2x+x^*B^2x)\le 2(a_1^2+b_1^2)$. As $c_1/\sqrt{2}=|a_1+ib_1|$, we have $C^*x=0$. Let y be a unit vector such that $y^*Cx=c_1$. Since $x^*C^*y=c_1$, we also have $\|C^*y\|=c_1$. As above, we deduce that $Cy=0$. As x and y are eigenvectors of C^*C corresponding to different eigenvalues, x and y are orthogonal. Let U be a unitary matrix with y and x as its first and second columns, respectively. Then we have $U^*CU = \begin{pmatrix} 0 & c_1 \\ 0 & 0\end{pmatrix} \oplus C'$. It is easy to check that C' satisfies the hypothesis and so we may repeat the same argument to conclude the result.

Case 3. $1<k<n$. An elegant proof of the weak majorization relation was given by Zhan [25]. Our study of equality cases follows his proof given by (in brief):

There exist $X,Y\in M_n$ such that $C=X+Y$ and $c_1+\cdots+c_k = s_1(X)+\cdots+s_n(X)+ks_1(Y)$. Let $X=P+iQ$ and $Y=E+iF$ be the Cartesian decompositions of X and Y, respectively. As the Cartesian decomposition is unique, we know that $A=P+E$ and $B=Q+F$. We have $\sqrt{2}(s_1(X)+\cdots+s_n(X)) \ge |s_1(P)+is_1(Q)|+\cdots+|s_n(P)+is_n(Q)|$ and $\sqrt{2}s_1(Y)\ge |s_1(E)+is_1(F)|$, and thus

$$\begin{aligned}\sqrt{2}(c_1+\cdots+c_k) &\ge \sum_{j=1}^n |s_j(P)+is_j(Q)| + k|s_1(E)+is_1(F)|\\ &\ge \sum_{j=1}^k |s_j(P)+is_j(Q)| + k|s_1(E)+is_1(F)| &(3.7)\end{aligned}$$

$$
\begin{aligned}
&= \sum_{j=1}^{k}(|s_j(P)+is_j(Q)|+|s_1(E)+is_1(F)|) \\
&\geq \sum_{j=1}^{k}|(s_j(P)+s_1(E))+i(s_j(Q)+s_1(F))| \\
&\geq \sum_{j=1}^{k}|a_j+ib_j|. \qquad (3.8)
\end{aligned}
$$

From our assumption, we know that all of these inequalities are equalities. In particular, we know that Y and X are of the forms that we deduced in cases 1 and 2. We now prove that either X or Y is the zero matrix. Suppose Y is nonzero. If X is nonzero then P and Q are nonzero. Equality in (3.7) implies that P and Q have rank at most k. From equality in (3.8), we have $s_j(P)+s_1(E)=a_j$, $j=1,\ldots,k$. It then follows from Proposition 1.2 that $s_1(E)=\cdots=s_k(E)$ and there exists a unitary matrix U such that $U^*PU=P_1\oplus P_2$, $U^*EU=E_1\oplus E_2$, where P_1 has singular values $s_1(P),\ldots,s_k(P)$ and E_1 has singular values $s_1(E),\ldots,s_k(E)$. As P has rank at most k, $P_2=0$. Equality in (3.8) also implies that $s_1(F)=\cdots=s_k(F)$. From $\sqrt{2}s_1(Y)=|s_1(E)+is_1(F)|$, an argument similar to our proof in Case 1 shows that $U^*FU=0_k\oplus F'$. Then, again using equality in (3.8), we deduce that $U^*QU=0_k\oplus Q'$ because Q has at most rank k. Then $U^*XU=U^*(P_1\oplus Q')U$, which implies that X^2 is nonzero. However, by Case 2, $X^2=0$ and this gives a contradiction. Thus $C=X$ or $C=Y$. Suppose $C=X$ and X has rank p. If equality holds, one easily checks that $2p\leq k$. Suppose $C=Y$. Notice that equality in (3.8) implies that $s_1(E)=\cdots=s_k(E)$ and $s_1(F)=\cdots=s_k(F)$, even when $X=0$. Applying Case 1 repeatedly gives the result. □

The inequalities (3.2) imply that $(c_1^2,\ldots,c_n^2)\prec_w 2(a_1^2+b_1^2,\ldots,a_n^2+b_n^2)$. Recently, it was proved in [4] that the constant 2 can be removed if A and B are positive semi-definite. We now study the equality cases.

Theorem 3.4 *Suppose A and B are $n\times n$ positive semi-definite matrices and $C=A+iB$. Then*

$$(c_1^2,\ldots,c_n^2)\prec(a_1^2+b_1^2,\ldots,a_n^2+b_n^2).$$

For any given $k\in\{1,\ldots,n\}$,

$$\sum_{j=1}^{k}c_j^2=\sum_{j=1}^{k}(a_j^2+b_j^2)$$

if and only if C is unitarily similar to $C_1\oplus C_2$, $s(C_1+C_1^*)=2(a_1,\ldots,a_k)$, and $s(C_1^*-C_1)=2(b_1,\ldots,b_k)$.

Proof. Since $\operatorname{tr}C^*C=\operatorname{tr}(A^2+B^2)$,

$$\sum_{j=1}^{k}c_j^2=\sum_{j=1}^{k}(a_j^2+b_j^2)$$

if and only if

$$\sum_{j=k+1}^{n} s_j(C^*C) = \sum_{j=k+1}^{n} (a_j^2 + b_j^2).$$

Suppose X is $n \times (n-k)$ and that its columns are orthonormal eigenvectors of C^*C corresponding to the eigenvalues $\lambda_j(C^*C) = s_j(C^*C)$ for $j = k+1, \ldots, n$. Then

$$\begin{aligned}
\sum_{j=k+1}^{n} s_j(C^*C) &= \operatorname{tr}(X^*C^*CX) \\
&\geq \operatorname{tr}(X^*C^*XX^*CX) \qquad (3.9)\\
&\geq \operatorname{tr}(X^*AXX^*AX) + \operatorname{tr}(X^*BXX^*BX) \\
&\qquad + i\operatorname{tr}[(X^*AX)(X^*BX) - (X^*BX)(X^*AX)] \\
&= \operatorname{tr}(X^*AXX^*AX) + \operatorname{tr}(X^*BXX^*BX) \\
&\geq \sum_{j=k+1}^{n} \{\lambda_j(A^2) + \lambda_j(B^2)\}, \qquad (3.10)
\end{aligned}$$

where inequality (3.9) follows from the fact that $I - XX^*$ is positive definite, and inequality (3.10) follows from the facts that $\lambda_j(X^*AX) \geq \lambda_{k+j}(A)$ and $\lambda_j(X^*BX) \geq \lambda_{k+j}(B)$ for $j = 1, \ldots, n-k$. Thus, equality holds in (3.10) if and only if X^*AX and X^*BX have eigenvalues $\lambda_j(A) = s_j(A)$ and $\lambda_j(B) = s_j(B)$ for $j = k+1, \ldots, n$. If U is unitary and its last $n-k$ columns are the columns of X, then $U^*AU = A_1 \oplus (X^*AX)$ and $U^*BU = B_1 \oplus (X^*BX)$. Hence U^*CU has the described form. □

Two other sets of majorization relations involving squares of singular values were obtained in [1] (see also [15]):

$$(c_1^2 + c_n^2, \ldots, c_n^2 + c_1^2)/2 \prec (a_1^2 + b_1^2, \ldots, a_n^2 + b_n^2), \qquad (3.11)$$

and

$$(a_1^2 + b_n^2, \ldots, a_n^2 + b_1^2) \prec (c_1^2, \ldots, c_n^2). \qquad (3.12)$$

The equality cases of (3.11) and (3.12) are more complicated as there might not be a unitary U such that both U^*AU and U^*BU are direct sums when equality holds. This can be seen from the following example.

Example 3.5 Let $A = \operatorname{diag}(1, -1)$, $B = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}$, and $C = A + iB$. Then for $k = 1$, equality holds in (3.11) and (3.12), but C is not normal and so it is not unitarily similar to a direct sum. In general, if $a_1 = \cdots = a_n$ and $b_1 = \cdots = b_n$, all the inequalities in the majorization (3.11) become equalities but C does not have any special reducibility structure.

In the following theorem, we need to impose additional conditions of the form $a_k > a_{k+1}$ and $b_k > b_{k+1}$ in order to study the equality cases of (3.11), and of the form $c_k > c_{k+1}$ in order to study those of (3.12). However, for (3.11), there are some cases in which we cannot impose such conditions for the following reasons. Let $(d_1,\dots,d_n) \equiv (1/2)(c_1^2+c_n^2,\dots,c_n^2+c_1^2)_\downarrow$. Suppose that $d_k = d_{k+1}$. If we assume that $d_1+\cdots+d_k = (a_1^2+b_1^2)+\cdots+(a_k^2+b_k^2)$, then we have $d_k \geq a_k^2+b_k^2$. Together with $d_1+\cdots+d_{k+1} \leq (a_1^2+b_1^2)+\cdots+(a_{k+1}^2+b_{k+1}^2)$, we must have $a_k = a_{k+1}$ and $b_k = b_{k+1}$. Thus we cannot assume either that $a_k > a_{k+1}$ or that $b_k > b_{k+1}$ if $d_k = d_{k+1}$. Notice that every element of the set of entries of $(c_1^2+c_n^2,\dots,c_n^2+c_1^2)/2$ has multiplicity at least two except possibly when n is odd and the entry is $c_{(n+1)/2}^2 + c_{(n+1)/2}^2$.

Theorem 3.6 *Suppose $A, B \in M_n$ are Hermitian and $C = A+iB$. Then (3.11) and (3.12) hold.*

(a) *Let $(d_1,\dots,d_n) \equiv (1/2)(c_1^2+c_n^2,\dots,c_n^2+c_1^2)_\downarrow$. Suppose that $k \in \{1,\dots,n-1\}$, $d_k > d_{k+1}$, $a_k > a_{k+1}$, and $b_k > b_{k+1}$. Then*

$$\sum_{j=1}^{k}(a_j^2+b_j^2) = \sum_{s=1}^{k}(c_{j_s}^2 + c_{n-j_s+1}^2)/2 \tag{3.13}$$

for some integers $j_1,\dots,j_k$ such that $1 \leq j_1 < \cdots < j_k \leq n$ if and only if C is unitarily similar to $C_1 \oplus C_2$ with $C_1 \in M_k$ such that $s((C_1+C_1^*)/2) = (a_1,\dots,a_k)$, $s((C_1-C_1^*)/2) = (b_1,\dots,b_k)$, and $\sum_{j=1}^k s_j(C_1C_1^*+C_1^*C_1) = \sum_{s=1}^k (c_{j_s}^2 + c_{n-j_s+1}^2)$.

(b) *Suppose that $k \in \{1,\dots,n-1\}$ and $c_k > c_{k+1}$. Then*

$$\sum_{j=1}^{k} c_j^2 = \sum_{s=1}^{k}(a_{j_s}^2 + b_{n-j_s+1}^2)$$

for some integers $j_1,\dots,j_k$ such that $1 \leq j_1 < \cdots < j_k \leq n$ if and only if C is unitarily similar to $C_1 \oplus C_2$, $C_1 \in M_k$ has singular values $c_1,\dots,c_k$, $s((C_1+C_1^*)/2) = (a_{j_1},\dots,a_{j_k})$, and $s((C_1-C_1^*)/2) = (b_{n-j_k+1},\dots,b_{n-j_1+1})$.

Proof. (a) ($\Leftarrow$) Direct verification.
($\Rightarrow$) Since $CC^*+C^*C = 2(A^2+B^2)$, by (1.3) and [24, Theorem 2], we have

$$\sum_{j=1}^{k}[s_j(A^2)+s_j(B^2)] \geq \sum_{j=1}^{k} s_j(A^2+B^2) = \sum_{j=1}^{k} s_j(CC^*+C^*C)/2 \geq \sum_{j=1}^{k}(c_{j_s}^2 + c_{n-j_s+1}^2)/2.$$

The equality (3.13) ensures that

$$\sum_{j=1}^{k} s_j(A^2+B^2) = \sum_{j=1}^{k}\left(s_j(A^2)+s_j(B^2)\right).$$

By Proposition 1.2 and the fact that A^2 and B^2 are positive semi-definite, we conclude that there exists a unitary matrix U such that $U^*A^2U = A_1 \oplus A_2$ and $U^*B^2U = B_1 \oplus B_2$ so that

$s(A_1) = (a_1^2, \ldots, a_k^2)$ and $s(B_1) = (b_1^2, \ldots, b_k^2)$. Thus, the span of the first k columns of U is a direct sum of eigenspaces of A corresponding to the eigenvalues $\lambda_j(A^2)$ for $j = 1, \ldots, k$. We see that $U^*AU = \tilde{A}_1 \oplus \tilde{A}_2$ is also in block form. Similarly, $U^*BU = \tilde{B}_1 \oplus \tilde{B}_2$. Since

$$\sum_{j=1}^{k} s_j((C_1C_1^* + C_1^*C_1)/2) = \sum_{j=1}^{k} s_j(A_1 + B_1) = \sum_{s=1}^{k} (c_{j_s}^2 + c_{n-j_s+1}^2),$$

the matrix $C_1 = \tilde{A}_1 + i\tilde{B}_1$ satisfies the specified condition.

(b) ($\Leftarrow$) Direct verification.

($\Rightarrow$) We have

$$\sum_{j=1}^{k} c_j^2 = \sum_{s=1}^{k} (a_{j_s}^2 + b_{n-j_s+1}^2) \le \sum_{j=1}^{k} \lambda_i((CC^* + C^*C)/2) \le \sum_{j=1}^{k} c_j^2.$$

Since the inequalities are equalities, Proposition 1.2 ensures that there exists a unitary matrix U such that $U^*C^*CU = D_1 \oplus D_2$, $U^*CC^*U = E_1 \oplus E_2$, and $s(D_1) = s(E_1) = (c_1^2, \ldots, c_k^2)$. Thus, the matrix formed by the first k rows (respectively, columns) of U^*CU has singular values $c_1 \ge \cdots \ge c_k$. Thus, there exist unitary matrices $V_1 \in M_k$ and $V_2 \in M_{n-k}$ such that $C_1 = (V_1 \oplus V_2)U^*CU$ has rows with ℓ_2 norm equal to $c_1 \ge \cdots \ge c_n$. Note that the matrix formed by the first k columns of C_1 still has singular values $c_1 \ge \cdots \ge c_k$. Thus, there exist unitary matrices $V_3 \in M_k$ and $V_4 \in M_{n-k}$ such that $C_2 = (V_1 \oplus V_2)U^*CU(V_3 \oplus V_4)$ has columns with ℓ_2 norm equal to $c_1 \ge \cdots \ge c_n$. In particular, $C_2 = DX = YD$ for some unitary matrices X and Y with $D = \operatorname{diag}(c_1, \ldots, c_n)$. By considering the Euclidean norms of the rows and columns of DX and YD, one sees that X and Y are direct sums of square blocks according to the multiplicities of the diagonal entries of D. One can now check that $(V_1 \oplus I_{n-k})U^*CU(V_1^* \oplus I_{n-k})$ has the desired form. $\square$

We conclude this section with the following open problems.

Problem 3.7 *Let $a, b, c \in \mathbf{R}^n$ be nonnegative vectors. Determine necessary and sufficient conditions on these vectors for the existence of Hermitian matrices A and B, and $C = A+iB$ satisfying*

(i) *$s(C) = c$ and $s(A) = a$, or*

(ii) *$s(C) = c$ and $s(B) = b$, or*

(iii) *$s(A) = a$, $s(B) = b$, and $s(C) = c$.*

We give partial results for these problems, including the 2×2 case, in the next section.

4 Eigenvalues and Singular Values

In this section, we consider relations among the eigenvalues of Hermitian matrices A and B and the singular values of the matrix $C = A + iB$. The inequalities in the following were proved in [8] and [22] and, following the proofs there, the equality cases can be easily verified.

Theorem 4.1 *Suppose $A, B \in M_n$ are Hermitian, $C = A + iB$, and $1 \leq j \leq n$.*

(ai) *If $\lambda_j(A) \geq 0$ then $\lambda_j(A) \leq s_j(C)$. Equality holds if and only if C is unitarily similar to $[\lambda_j(A)] \oplus C_1$ with $\lambda_j(A) \geq s_j(C_1)$.*

(aii) *If $\lambda_j(A) \leq 0$ then $\lambda_j(A) \leq s_{n-j+1}(C)$. Equality holds if and only if C is unitarily similar to $[\lambda_j(A)] \oplus C_1$ with $|\lambda_j(A)| \geq s_{n-j+1}(C_1)$.*

(bi) *If $\lambda_j(B) \geq 0$ then $\lambda_j(B) \leq s_j(C)$. Equality holds if and only if C is unitarily similar to $[i\lambda_j(B)] \oplus C_1$ with $\lambda_j(B) \geq s_j(C_1)$.*

(bii) *If $\lambda_j(B) \leq 0$ then $\lambda_j(B) \leq s_{n-j+1}(C)$. Equality holds if and only if C is unitarily similar to $[i\lambda_j(B)] \oplus C_1$ with $|\lambda_j(B)| \geq s_{n-j+1}(C_1)$.*

Problem 4.2 *Let $a, b, c \in \mathbf{R}^n$, and suppose c has nonnegative entries. Determine necessary and sufficient conditions for the existence of Hermitian matrices A and B, and $C = A + iB$ satisfying*

(i) *$s(C) = c$ and $\lambda(A) = a$, or*

(ii) *$s(C) = c$ and $\lambda(B) = b$, or*

(iii) *$\lambda(A) = a$, $\lambda(B) = b$, and $s(C) = c$.*

Theorem 4.1 gives the following partial answer for this problem, which is also a partial answer to Problem 3.7.

Theorem 4.3 *Suppose $a, b, c \in \mathbf{R}^n$, and suppose that c has nonnegative entries.*

(i) *Suppose a is nonnegative. Then there exist Hermitian matrices A and B such that $C = A + iB$, $s(C) = c$, and $\lambda(A) = a$ if and only if $c_\downarrow - a_\downarrow$ is nonnegative.*

(ii) *Suppose b is nonnegative. Then there exist Hermitian matrices A and B such that $C = A + iB$, $s(C) = c$, and $\lambda(B) = b$ if and only if $c_\downarrow - b_\downarrow$ is nonnegative.*

The general case seems much more difficult. Even for the 2×2 case, the answer is non-trivial; see [15], where the solution given is not in terms of linear inequalities.

Theorem 4.4 *Let $c_1, c_2, \alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbf{R}$ be such that $c_1 \geq c_2 \geq 0$, $|\alpha_1| \geq |\alpha_2|$, and $|\beta_1| \geq |\beta_2|$. Then there exist Hermitian matrices A and B such that $C = A + iB \in M_2$, $s(C) = (c_1, c_2)$, and A and B have eigenvalues α_1, α_2 and β_1, β_2, respectively, if and only if*

$$(\alpha_1^2 + \beta_2^2, \alpha_2^2 + \beta_1^2) \prec (c_1^2, c_2^2), \tag{4.1}$$

and

$$(c_1c_2)^2 - ((\alpha_1\alpha_2) - (\beta_1\beta_2))^2 \geq \max\left\{0, \delta(\alpha_1\beta_2 + \alpha_2\beta_1)^2\right\}, \tag{4.2}$$

where δ is the sign of $(\alpha_1\beta_1 + \alpha_2\beta_2)(\alpha_1\beta_2 + \alpha_2\beta_1)$. Consequently, there exist Hermitian matrices A and B such that $C = A + iB \in M_2$, $s(C) = (c_1, c_2)$, $s(A) = (|\alpha_1|, |\alpha_2|)$, and $s(B) = (|\beta_1|, |\beta_2|)$ if and only if (4.1) holds and (4.2) holds with $\delta = -1$.

5 Determinantal Inequalities

In this section, we study determinantal inequalities involving Hermitian matrices A and B, and $C = A + iB$. We begin with the following observation.

Proposition 5.1 *Suppose $A, B \in M_n$ are Hermitian matrices such that A is positive definite. Then*

$$\det(A + iB) = \det(A) \prod_{j=1}^{n} \left(1 + s_j(A^{-1/2}BA^{-1/2})^2\right)^{1/2}.$$

Using this observation, Thompson [22] (see also [8]) proved the following interesting result.

Theorem 5.2 *Suppose $A, B \in M_n$ are Hermitian and $C = A + iB$. Then*

$$|\det(C)|^{2/n} \geq R^{1/n}|\det(A)|^{2/n} + |\det(B)|^{2/n},$$

where the real constant R is arbitrary when A, B, C are all singular; otherwise,

$$R = \prod_t \frac{|\xi_t^2 + 1|}{|\xi_t|^2 + 1}, \tag{5.1}$$

where the product extends over all nonreal roots ξ_t of the equation $\det(\lambda A - B) = 0$. *Equality holds in* (5.1) *if and only if* (a) *all the matrices A, B, and C are singular, or* (b) *all roots of* $\det(\lambda A - B) = 0$ *have equal modulus.*

The following inequality was proved in [1]. We study the equality case.

Theorem 5.3 *Suppose $A, B \in M_n$ are Hermitian and $C = A + iB$. Then*

$$|\det(C)| \leq \prod_{j=1}^{n} |s_j(A) + is_{n-j+1}(B)|. \tag{5.2}$$

Equality holds in (5.2) *if and only if*

(a) $\operatorname{rank}(A) + \operatorname{rank}(B) < n$, *or*

(b) C *is unitarily similar to* $\operatorname{diag}(\gamma_1, \ldots, \gamma_n)$, *with* $\gamma_j = |s_j(A) + is_{n-j+1}(B)|$ *for all* j.

Proof. We focus on the equality case. The sufficiency part is clear. Suppose equality holds. If (a) is not true, then because

$$(s_1(A)^2 + s_n(B)^2, \ldots, s_n(A)^2 + s_1(B)^2) \prec (s_1(C)^2, \ldots, s_n(C)^2)$$

and the products of the entries of the two vectors are equal, [19, Chapter 3, Proposition F.1] ensures that the vectors have the same entries up to a permutation.

If $s_j(C)$ are equal for all j, then C is unitary and condition (b) holds. If not all $s_j(C)$ are equal, let k be the smallest integer such that $s_k(C) > s_{k+1}(C)$. By Theorem 3.6 (b), we see that C is unitarily similar to $\gamma_1 U_1 \oplus C_2$, where $\gamma_1 = s_1(C)$ and $U_1 \in M_k$ is unitary. One can apply an inductive argument to C_2 to conclude that C is unitarily similar to $\gamma_1 U_1 \oplus \cdots \oplus \gamma_m U_m$, where $\gamma_1 \geq \cdots \geq \gamma_m \geq 0$ are the distinct singular values of C, and $U_1, \ldots, U_m$ are unitary. One easily checks that condition (b) holds. $\square$

We also have the following (see also [4, Theorem 3]).

Theorem 5.4 *Let* $A, B \in M_n$ *be positive semi-definite and* $C = A + iB$. *For* $1 \leq k \leq n$, *we have*

$$\prod_{j=k}^{n} |s_j(C)| \geq \prod_{j=k}^{n} |s_j(A) + is_j(B)|. \tag{5.3}$$

Equality holds in (5.3) *if and only if* C *is unitarily similar to*

$$C_1 \oplus \operatorname{diag}(s_k(A) + is_k(B), \ldots, s_n(A) + is_n(B)).$$

Proof. By Theorem 3.4,

$$\sum_{j=p}^{n} |s_j(C)| \geq \sum_{j=p}^{n} |s_j(A) + is_j(B)|$$

for $p = 1, \ldots, n$. The inequality (5.3) now follows from [19, Chapter 3, Proposition E.1]. Equality holds in (5.3) if and only if $s_j(C) = |(s_j(A) + is_j(B)|$ for $j = k, \ldots, n$. Thus C is unitarily similar to

$$C_1 \oplus \operatorname{diag}(s_k(A) + is_k(B), \ldots, s_n(A) + is_n(B)).$$

The converse is easy to verify. $\square$

Note that the inequality (5.3) need not hold if one of A or B is not positive semi-definite.

Example 5.5 Let $C = A + iB$ with

$$A = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad B = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Then $s_2(C) = (\sqrt{5} - 1)/2 < 1 = |s_2(A) + is_2(B)|$ and $s_1(C)s_2(C) = 1 < \sqrt{2} = |(s_1(A) + is_1(B))(s_2(A) + is_2(B))|$.

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