INVERSE CLOSED RAY-NONSINGULAR CONES OF MATRICES

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Abstract. A description is given of those ray-patterns, which will be called inverse closed ray-nonsingular, of complex matrices that contain invertible matrices only and are closed under inversion. Here, two $n \times n$ matrices are said to belong to the same ray-pattern if in each position either the entries of both matrices are zeros, or both entries are nonzero and their quotient is positive. Possible Jordan forms of matrices in the inverse closed ray-nonsingular ray-patterns are characterized.

1. Introduction

In this paper we study ray-patterns of complex matrices with the property that every matrix in the ray-pattern is nonsingular, and the inverse of the matrix is again in the same ray-pattern.

We first introduce some terminology and notation. Let

$$\Omega = \{0\} \cup \{z \in \mathbb{C} : |z| = 1\}.$$

A matrix with entries in Ω will be called a *ray-pattern*. Denote by $M_n(\Omega)$ the set of $n \times n$ ray-patterns. A pattern $A \in M_n(\Omega)$ generates a cone $\mathrm{Cone}\,(A)$ in a natural way:

$$\mathrm{Cone}\,(A) = \{X \circ A : X \;\; n \times n \text{ matrix with positive entries}\},$$

where $\circ$ stands for the Hadamard (entrywise) product.

A ray-pattern $A \in M_n(\Omega)$ is called *ray-nonsingular* if $X \circ A$ is invertible for every $n \times n$ matrix X with positive entries. Ray-patterns, in particular ray-nonsingular ray-patterns, were studied recently in [6], [8], [5], For a survey of the theory of *real* ray-nonsingular ray-patterns see the book [1] and references there.

In applications of matrix analysis to stability (see, e.g., [2], [3]) one often encounters closed cones of matrices with the property that the set of invertible matrices in the cone is dense, and the inverse of every invertible matrix in the cone belongs again to the cone. In the context of ray-nonsingular ray-patterns such cones appear as follows. We denote by $M_n(\mathbb{C})$ the algebra of complex $n \times n$ matrices. Given $X = [x_{i,j}]_{i,j=1}^n \in M_n(\mathbb{C})$, define the *pattern projection* $\mathrm{ray}\,(X)$ of X as follows: $\mathrm{ray}\,(X)$ is the ray-pattern whose (i,j)-th entry is 0 if $x_{i,j} = 0$, and is equal to $\frac{x_{i,j}}{|x_{i,j}|}$ if $x_{i,j} \neq 0$. A ray-nonsingular ray-pattern A is called *inverse-closed* if $\mathrm{ray}\,((X \circ A)^{-1}) = A$ for every matrix X with positive entries.

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Lemma 1.1. *Suppose A is an inverse-closed ray-nonsingular ray-pattern. Denote by* $\mathrm{Cone}\,(A)$ *the cone generated by A, and let* $\mathrm{CCone}\,(A)$ *be the closed cone which is the closure of* $\mathrm{Cone}\,(A)$*:*

$$\mathrm{CCone}\,(A) = \{X \circ A \,:\, X \quad \textit{has nonnegative entries}\}.$$

Then every $Y \in \mathrm{Cone}\,(A)$ is invertible, and every invertible $Y \in \mathrm{CCone}\,(A)$ has the property that $Y^{-1} \in \mathrm{CCone}\,(A)$.

Proof. Let $Y = X \circ A$, for some X with nonnegative entries, and assume that Y is invertible. Let X_m be a sequence of matrices with positive entries such that $X_m \longrightarrow X$ as $m \longrightarrow \infty$. Then clearly $Y_m := X_m \circ A$ is invertible for large m, and $Y_m^{-1} \longrightarrow Y^{-1}$. By the properties of inverse-closeness we have that $Y_m^{-1} \in \mathrm{Cone}\,(A)$. Passing to the limit we obtain $Y^{-1} \in \mathrm{CCone}\,(A)$, as required. □

In this paper, we solve the following problem.

Problem 1.2. *Describe all inverse-closed ray-nonsingular (in short, ICRN) ray-patterns.*

A solution of the problem is known in the real case, see [1, Section 7.4], [4]. A closely related (but different) problem, also in the real case, was studied in [10], [11], [12]. Another relevant paper is [8].

The ICRN ray-patterns transform well with respect to certain similarity transformations. Let $\mathcal{G}_n$, sometimes abbreviated to $\mathcal{G}$ if the size n is understood from context, be the group of matrices that consists of all products of the form DQ, where Q is an $n \times n$ permutation matrix and D is a *unimodular diagonal matrix*, i.e., diagonal matrix with unimodular entries on the main diagonal. Clearly, gA and Ag are ray-patterns for every $n \times n$ ray-pattern A and every $g \in \mathcal{G}_n$.

Lemma 1.3. *If a ray-pattern $A \in M_n(\Omega)$ is ICRN, then so are $\pm\overline{A}$ and $\pm A^t$ (the transpose of A), and every ray-pattern of the form gAg^{-1}, where $g \in \mathcal{G}_n$ is arbitrary. Furthermore, if $A_1, \ldots, A_p$ are ICRN's, then so is* $\mathrm{diag}\,(A_1, \ldots, A_p)$.

We omit a simple proof.

In view of Lemma 1.3, we will describe the ICRN ray-patterns up to a similarity with the similarity matrix in $\mathcal{G}_n$.

The rest of the paper consists of 5 sections. Sections 2 and 3 are preparatory, and contain descriptions of ICRN ray-patterns without zero entries and irreducible ICRN ray-patterns. The main result of the paper, Theorem 4.2, is stated in Section 4. Its rather long proof is delegated to Section 5. In Section 6 we study location of eigenvalues and inertia properties of matrices in the cone $\mathrm{Cone}\,(A)$, for ICRN ray-patterns A. In particular, we completely describe Jordan forms of matrices in the set $\cup_A\,(\mathrm{Cone}\,(A))$, where the union is taken over all $n \times n$ ICRN ray-patterns A.

2. ICRN ray-patterns without zero entries

Lemma 2.1. *Let A be an $n \times n$ ray-nonsingular ray-pattern without zero entries such that the ray-pattern projection* $\operatorname{ray}((X \circ A)^{-1})$ *is independent of any matrix X with positive entries and all entries of* $\operatorname{ray}((X \circ A)^{-1})$ *are nonzero. Then $n \leq 2$, and there exist $g, h \in \mathcal{G}_n$ such that gAh has the form*

$$[1] \quad \text{or} \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Proof. It is known that there do not exist ray-nonsingular $n \times n$ ray-patterns without zero entries if $n > 4$ [6], [7]. We will focus on the cases for $n = 1, \ldots, 4$, and for completeness present also the case $n = 5$.

The hypotheses on A are clearly invariant under pre- and post-multiplication by elements of $\mathcal{G}_n$. Using a transformation of the form $A \mapsto gAh$, $g, h \in \mathcal{G}$, we may assume that the entries of A in the first row and the first column are all one, viz.,

$$A = [z_{i,j}]_{i,j=1}^n \qquad \text{with } \ |z_{i,j}| = 1, \ \text{ and } z_{i,j} = 1 \ \text{ whenever } 1 \in \{i, j\}.$$

Then the result is trivial if $n = 1$. Suppose $n = 2$. Let

$$X = \begin{bmatrix} 1 & 1 \\ 1 & r \end{bmatrix}$$

with $r > 0$. Then the $(1,1)$ entry of $X \circ A$ is $z_{2,2}/(rz_{2,2} - 1)$ which has the same argument for any $r > 0$. Thus, $rz_{2,2} - 1$ has the same argument for all $r > 0$, and hence $z_{2,2} = -1$, i.e., A has the asserted form.

Consider the case $n = 3$. Let

$$X = \begin{bmatrix} 1 & u & v \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

Then the $(3,3)$ entry and $(2,3)$ entry of $(X \circ A)^{-1}$ are

$$(z_{2,2} - u)/\det(X \circ A) \qquad \text{and} \qquad (v - z_{2,3})/\det(X \circ A), \tag{2.1}$$

respectively, and each of them has a fixed argument for all choices of $u, v > 0$. Dividing the numbers (2.1), we see that $(z_{2,2} - u)/(v - z_{2,3})$ has a fixed argument for all choices of $u, v > 0$. Fixing v and changing u, we see that $z_{2,2} = -1$; fixing u and changing v, we see that $z_{2,3} = -1$. Now, applying the argument to $\tilde{A}$, where $\tilde{A}$ is obtained from A by interchanging its last two rows, we see that $z_{3,2} = z_{3,3} = -1$. But then

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & -1 \\ 1 & -1 & -1 \end{bmatrix}$$

is not ray-nonsingular, which is a contradiction.

Consider the case $n = 4$. We may assume that $z_{2,2}z_{3,k} - z_{3,2}z_{2,k} \neq 0$ for some $k \geq 3$; otherwise, the $(1,4)$ entry of A^{-1} is zero, which is excluded by the hypotheses. We may assume that $k = 3$; otherwise, interchange the last two columns of A. Let

$$X = \begin{bmatrix} 1 & r & s & 1 \\ t & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \qquad \text{with} \quad r, s, t > 0.$$

Then the $(4,4)$ entry and the $(1,1)$ entry of $(X \circ A)^{-1}$ have constant (i.e., independent of r, s, t) arguments, and so is their quotient:

$$\frac{(z_{2,2}z_{3,3} - z_{2,3}z_{3,2}) - r(tz_{3,3} - z_{2,3}) + s(tz_{3,2} - z_{2,2})}{z_{2,2}z_{3,3}z_{4,4} + z_{2,4}z_{3,2}z_{4,3} + z_{4,2}z_{2,3}z_{3,4} - z_{4,2}z_{3,3}z_{2,4} - z_{3,2}z_{2,3}z_{4,4} - z_{4,3}z_{3,4}z_{2,2}}.$$

As a result, for any $t > 0$ not equal to $z_{2,2}/z_{3,2}$ or $z_{2,3}/z_{3,3}$, the quantity

$$(z_{2,2}z_{3,3} - z_{2,3}z_{3,2}) - r(tz_{3,3} - z_{2,3}) + s(tz_{3,2} - z_{2,2})$$

has a constant argument for any $r, s > 0$. Hence, $z_{2,3} - tz_{3,3}$ and $tz_{3,2} - z_{2,2}$ have the same argument as $z_{2,2}z_{3,3} - z_{2,3}z_{3,2}$ for all $t > 0$ not equal to $z_{2,2}/z_{3,2}$ or $z_{2,3}/z_{3,3}$. It follows that $z_{3,3} = -z_{2,3}$ and $z_{3,2} = -z_{2,2}$. But then $z_{2,2}z_{3,3} - z_{2,3}z_{3,2} = 0$, which is a contradiction.

Now, suppose $n = 5$. Let

$$X = \begin{bmatrix} 1 & r & s & t & 1 \\ u & x & 1 & 1 & 1 \\ v & 1 & 1 & 1 & 1 \\ w & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \qquad \text{with} \quad r, s, t, u, v, w, x > 0.$$

Denoted by $Y(p, q)$ the matrix obtained from the matrix Y by removing its pth row and qth column. Let $B = (X \circ A)(5, 5)$. Then the $(5, 5)$ and $(1, 1)$ entries of $(X \circ A)^{-1}$ have constant arguments and so does their quotient, which has the form

$$\frac{\det(B(1,1)) - r\det(B(1,2)) + s\det(B(1,3)) - t\det(B(1,4))}{\det(X \circ A)(1,1)}. \tag{2.2}$$

Note that $B(1, 1)$ and $A(1, 1)$ depend only on the variable x; $B(1, 2), B(1, 3), B(1, 4)$ depend on the variables u, v, w, x. For fixed $x > 0$, the quotient (2.2) has constant argument for all choices of $r, s, t, u, v, w > 0$. It follows that

$$\det(B(1,3)) = u(z_{3,2}z_{4,4} - z_{4,2}z_{3,4}) - v(xz_{2,2}z_{4,4} - z_{4,2}z_{2,4}) + w(xz_{2,2}z_{3,4} - z_{3,2}z_{2,4}) \tag{2.3}$$

is either zero or has a constant argument for all choices of $u, v, w > 0$ if $x > 0$ is fixed. Choosing $x > 0$ such that both $xz_{2,2}z_{4,4} - z_{4,2}z_{2,4}$ and $xz_{2,2}z_{3,4} - z_{3,2}z_{2,4}$ are nonzero, and varying $u, v, w > 0$ in (2.3), we see that $xz_{2,2}z_{4,4} - z_{4,2}z_{2,4}$ and $xz_{2,2}z_{3,4} - z_{3,2}z_{2,4}$ always have the same arguments. Hence, they are positive multiples of each other. This is true for infinitely many $x > 0$. It follows that

$$z_{4,4}/z_{4,2} = z_{3,4}/z_{3,2}.$$

Interchange the kth row and the third row of A for $k = 2, 5$, and repeat the above argument. We conclude that

$$z_{4,4}/z_{4,2} = z_{k,4}/z_{k,2}, \qquad k = 2, 3, 5.$$

Thus, the second column and the fourth column of $A(1,1)$ are multiples of each other. It follows that $\det(A(1,1)) = 0$, which is a contradiction. Our proof is complete. □

Proposition 2.2. *An $n \times n$ ray-pattern A without zero entries is ICRN if and only if $n \le 2$ and there exists $g \in \mathcal{G}_n$ such that gAg^{-1} has one of the three forms*

$$[1], \qquad [-1], \qquad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Proof. The "if" part is obvious. For the "only if" part, by Lemma 2.1 we have $n \le 2$. The case $n = 1$ is trivial. For $n = 2$, we may apply a transformation $A \mapsto gAg^{-1}$, $g \in \mathcal{G}$ and assume that

$$A = \begin{bmatrix} 1 & 1 \\ v & w \end{bmatrix},$$

where v, w are unimodular numbers. Let

$$X = \begin{bmatrix} 1 & 1 \\ x & 1 \end{bmatrix} \qquad \text{with } \; x > 0,$$

and consider the $(1,1)$ and $(2,1)$ entries of $(X \circ A)^{-1}$. We see that $w/(w - xv) > 0$ and $-1/(w - xv) > 0$ for all $x > 0$. Thus, $v = -w = 1$. □

3. Irreducible ICRN ray-patterns

A ray-pattern A is called *irreducible* if there is no permutation matrix Q such that

$$QAQ^{-1} = \begin{bmatrix} A_{1,1} & 0 \\ A_{2,1} & A_{2,2} \end{bmatrix},$$

where the sizes of the square submatrices $A_{1,1}$ and $A_{2,2}$ are strictly smaller than that of A.

Proposition 3.1. *Let A be an $n \times n$ irreducible ray-pattern. Then A is ICRN if and only if $n \in \{1, 2, 4\}$ and there exists $g \in \mathcal{G}_n$ such that gAg^{-1} has one of the following five forms:*

$$[1], \quad [-1], \quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}. \tag{3.1}$$

Proof. The proof follows the argument from [1, Section 7.4]. The "if" part is clear. Assume A is an ICRN ray-pattern. First suppose that A is fully indecomposable, i.e., no $p \times q$ submatrix of A with $p + q \ge n$ is the zero matrix. Consider an $(n-1) \times (n-1)$ submatrix B of A. Since the ray-pattern projection ray $((X \circ A)^{-1})$ is independent of the positive matrix X, we have either $\det(X \circ B) = 0$ for every positive $(n-1) \times (n-1)$

matrix X, or $\det(X \circ B) \neq 0$ for every positive $(n-1) \times (n-1)$ matrix X. In the former case, all $(n-1)!$ terms in the expression of $\det B$ are zeros, which implies (as can be seen by induction on the size of B, for example) that there exists an $r \times s$ zero submatrix of B with $r + s > n - 1$ (the Frobenius - König theorem, see, e.g., [9]). This contradicts the full indecomposability of A. Thus, the latter case holds, which implies that A has no zero entries. Now use Proposition 2.2.

Suppose that A is not fully indecomposable. We denote by $Y[\alpha, \beta]$ the $|\alpha| \times |\beta|$ submatrix of an $n \times n$ matrix Y defined by the nonempty index set $\alpha \subseteq \{1, 2, \ldots, n\}$ of rows and the nonempty index set $\beta \subseteq \{1, 2, \ldots, n\}$ of columns; $|\alpha|$ stands for the cardinality of a set α. Let α and β be such that $|\alpha| + |\beta| = n$ and $A[\alpha, \beta] = 0$. Then for every positive matrix X we have $(X \circ A)^{-1}[\overline{\beta}, \overline{\alpha}] = 0$, where $\overline{\alpha}$ stands for the complement of α in $\{1, 2, \ldots, n\}$. Since A is ICRN, it follows that $A[\overline{\beta}, \overline{\alpha}] = 0$. If $\alpha \setminus \beta \neq \emptyset$, then $A[\alpha \setminus \beta, \overline{\alpha \setminus \beta}] = 0$, a contradiction with irreducibility of A. Thus, $\alpha \setminus \beta = \emptyset$. Similarly, $\beta \setminus \alpha = \emptyset$. So $\alpha = \beta$. Since

$$A[\alpha, \alpha] = 0, \quad A[\overline{\alpha}, \overline{\alpha}] = 0,$$

we conclude that n is even and $|\alpha| = n/2$.

If $A[\alpha, \overline{\alpha}]$ is not fully indecomposable, then there exists a $|\gamma| \times |\delta|$ zero submatrix $A[\gamma, \delta]$ with $|\gamma| + |\delta| = n$ and $\gamma \neq \delta$, a contradiction with what was proved above. Thus, $A[\alpha, \overline{\alpha}]$ and similarly $A[\alpha, \overline{\alpha}]$ are fully indecomposable. Applying a similarity $A \mapsto QAQ^{-1}$, with a permutation matrix Q, we may assume that

$$A = \begin{bmatrix} 0 & A_1 \\ A_2 & 0 \end{bmatrix},$$

where A_1 and A_2 are fully indecomposable $(n/2) \times (n/2)$ ray-patterns. Clearly,

$$\text{ray}\left((X \circ A_1)^{-1}\right) = A_2 \quad \text{and} \quad \text{ray}\left((X \circ A_2)^{-1}\right) = A_1$$

for every positive matrix X. In particular, the ray-pattern projections $\text{ray}\left((X \circ A_j)^{-1}\right)$ are independent of X, for $j = 1, 2$. The first paragraph of the proof shows that A_j have no zero entries, and an application of Lemma 2.1 completes the proof. $\square$

4. Reducible ICRN ray-patterns: the main result

Lemma 4.1. *Let*

$$A = \begin{bmatrix} A_{1,1} & 0 \\ A_{2,1} & A_{2,2} \end{bmatrix}$$

be an ICRN $n \times n$ ray-pattern, where $A_{1,1}$ and $A_{2,2}$ are irreducible ICRN ray-patterns. Then either $A_{2,1} = 0$ or there exists $g \in \mathcal{G}_n$ such that gAg^{-1} has one of the following forms:

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} -1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & -1 & 1 \end{bmatrix}, \tag{4.1}$$

$$(4.2)\qquad \begin{bmatrix} -1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & -1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & z & 0 & 1 \\ -z & -1 & 1 & 0 \end{bmatrix} \quad z \in \Omega.$$

Conversely, all matrices in (4.1) *and* (4.2) *are ICRN ray-patterns.*

Proof. The converse statement is verified in a straightforward way.

Let A be as in the lemma. By Proposition 3.1, each of the matrices $A_{1,1}$ and $A_{2,2}$ has one of the forms (3.1). We will prove that if at least one of $A_{1,1}$ and $A_{2,2}$ has either the third or the fifth form of (3.1), then $A_{2,1} = 0$.

Let

$$A_{1,1} = A_{2,2} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad A_{2,1} = \begin{bmatrix} z_{1,1} & z_{1,2} \\ z_{2,1} & z_{2,2} \end{bmatrix}.$$

Then for $x_{1,1}$, $x_{1,2}$, $x_{2,1}$, $x_{2,2}$ positive we have

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \\ x_{1,1}z_{1,1} & x_{1,2}z_{1,2} & 1 & 1 \\ x_{2,1}z_{2,1} & x_{2,2}z_{1,2} & 1 & -1 \end{bmatrix}^{-1} = \begin{bmatrix} A_{1,1}^{-1} & 0 \\ T & A_{2,2}^{-1} \end{bmatrix},$$

where

$$T = -A_{2,2}^{-1} \begin{bmatrix} x_{1,1}z_{1,1} & x_{1,2}z_{1,2} \\ x_{2,1}z_{2,1} & x_{2,2}z_{1,2} \end{bmatrix} A_{1,1}^{-1}.$$

The matrix T is computed to be

$$T = \frac{-1}{4} \begin{bmatrix} x_{1,1}z_{1,1} + x_{2,1}z_{2,1} + x_{1,2}z_{1,2} + x_{2,2}z_{2,2} & x_{1,1}z_{1,1} + x_{2,1}z_{2,1} - x_{1,2}z_{1,2} - x_{2,2}z_{2,2} \\ x_{1,1}z_{1,1} - x_{2,1}z_{2,1} + x_{1,2}z_{1,2} - x_{2,2}z_{2,2} & x_{1,1}z_{1,1} - x_{2,1}z_{2,1} - x_{1,2}z_{1,2} + x_{2,2}z_{2,2} \end{bmatrix}.$$

Since A is ICRN, we must have in particular that each of the entries of T has the same argument (or is zero) irrespectively of the positive values of $x_{1,1}$, $x_{1,2}$, $x_{2,1}$, $x_{2,2}$. It is easy to see that this happens only if at most one number among $z_{1,1}$, $z_{1,2}$, $z_{2,1}$, $z_{2,2}$ is nonzero. But if one of those numbers were nonzero, we would have $\operatorname{ray}(T) \neq A_{2,1}$, a contradiction with A being ICRN.

Let

$$A_{1,1} = \begin{bmatrix} 0 & Q \\ Q & 0 \end{bmatrix}, \quad A_{2,2} = Q,$$

where $Q = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$. Note that $Q^{-1} = \frac{1}{2}Q$. Let

$$A_{2,1} = [z_{i,j}]_{i=1,2;j=1,2,3,4}, \qquad X = [x_{i,j}]_{i=1,2;j=1,2,3,4}, \quad x_{i,j} > 0.$$

Then

$$\begin{bmatrix} A_{1,1} & 0 \\ X \circ A_{2,1} & A_{2,2} \end{bmatrix}^{-1} = \begin{bmatrix} A_{1,1}^{-1} & 0 \\ T & A_{2,2}^{-1} \end{bmatrix},$$

where

$$T = \frac{-1}{4} \left[Q \begin{bmatrix} x_{1,3}z_{1,3} & x_{1,4}z_{1,4} \\ x_{2,3}z_{2,3} & x_{2,4}z_{2,4} \end{bmatrix} Q \quad Q \begin{bmatrix} x_{1,1}z_{1,1} & x_{1,2}z_{1,2} \\ x_{2,1}z_{2,1} & x_{2,2}z_{2,2} \end{bmatrix} Q \right].$$

As in the preceding case, where $A_{1,1}$ and $A_{2,2}$ were both equal to Q, it follows (since ray (T) should be independent of X) that at most one number among $z_{1,1}$, $z_{1,2}$, $z_{2,1}$, $z_{2,2}$ is nonzero, and at most one number among $z_{1,3}$, $z_{1,4}$, $z_{2,3}$, $z_{2,4}$ is nonzero. Say, $z_{1,1} \neq 0$. The right 2×2 submatrix of T is

$$\frac{-1}{4} \begin{bmatrix} x_{1,1}z_{1,1} & x_{1,1}z_{1,1} \\ x_{1,1}z_{1,1} & x_{1,1}z_{1,1} \end{bmatrix}.$$

Since A is ICRN, we must have

$$P\left(\frac{-1}{4} \begin{bmatrix} x_{1,1}z_{1,1} & x_{1,1}z_{1,1} \\ x_{1,1}z_{1,1} & x_{1,1}z_{1,1} \end{bmatrix}\right) = \begin{bmatrix} z_{1,3} & z_{1,4} \\ z_{2,3} & z_{2,4} \end{bmatrix},$$

a contradiction with the property that at least three numbers among $z_{1,3}$, $z_{1,4}$, $z_{2,3}$, $z_{2,4}$ are zeros. Thus, $A_{2,1} = 0$. The cases when

$$A_{1,1} = A_{2,2} = \begin{bmatrix} 0 & Q \\ Q & 0 \end{bmatrix} \quad \text{or} \quad A_{1,1} = Q, \quad A_{2,2} = \begin{bmatrix} 0 & Q \\ Q & 0 \end{bmatrix},$$

are treated in a similar manner.

Let now consider the case

$$A_{1,1} = \begin{bmatrix} 0 & Q \\ Q & 0 \end{bmatrix}, \quad A_{2,2} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let

$$A_{2,1} = [z_{i,j}]_{i=1,2;j=1,2,3,4}, \qquad X = [x_{i,j}]_{i=1,2;j=1,2,3,4}, \quad x_{i,j} > 0.$$

We have

$$\begin{bmatrix} A_{1,1} & 0 \\ X \circ A_{2,1} & A_{2,2} \end{bmatrix}^{-1} = \begin{bmatrix} A_{1,1}^{-1} & 0 \\ T & A_{2,2}^{-1} \end{bmatrix},$$

where

$$T = \frac{1}{2} \begin{bmatrix} x_{2,3}z_{2,3} + x_{2,4}z_{2,4} & x_{2,3}z_{2,3} - x_{2,4}z_{2,4} & x_{2,1}z_{2,1} + x_{2,2}z_{2,2} & x_{2,1}z_{2,1} - x_{2,2}z_{2,2} \\ x_{1,3}z_{1,3} + x_{1,4}z_{1,4} & x_{1,3}z_{1,3} - x_{1,4}z_{1,4} & x_{1,1}z_{1,1} + x_{1,2}z_{1,2} & x_{1,1}z_{1,1} - x_{1,2}z_{1,2} \end{bmatrix}.$$

For ray (T) to be independent of $x_{i,j}$ we must have that there is at least one zero in each of the four pairs $\{z_{1,1}, z_{1,2}\}$, $\{z_{1,3}, z_{1,4}\}$, $\{z_{2,1}, z_{2,2}\}$, and $\{z_{2,3}, z_{2,4}\}$. But if one of the $z_{i,j}$ is nonzero, say $z_{1,1} \neq 0$, then the ray-pattern of the right lower 1×2 corner of T is not equal to $\begin{bmatrix} z_{2,3} & z_{2,4} \end{bmatrix}$, a contradiction with the ICRN property of A. The remaining cases of one of the two blocks $A_{1,1}$ and $A_{2,2}$ being equal to either Q or $\begin{bmatrix} 0 & Q \\ Q & 0 \end{bmatrix}$, and the other block being one of $[\pm 1]$ or $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ can be dealt with similarly.

Thus, leaving aside the case when $A_{2,1} = 0$, each of the matrices $A_{1,1}$ and $A_{2,2}$ has one of the first, second, or fourth forms of (3.1). The rest is elementary. For example,

$$\begin{bmatrix} 0 & y_2 & 0 & 0 \\ y_1 & 0 & 0 & 0 \\ y_5z_5 & y_6z_6 & 0 & y_4 \\ y_7z_7 & y_8z_8 & y_3 & 0 \end{bmatrix}^{-1} = \begin{bmatrix} 0 & y_1^{-1} & 0 & 0 \\ y_2^{-1} & 0 & 0 & 0 \\ -y_3^{-1}y_2^{-1}y_8z_8 & -y_3^{-1}y_1^{-1}y_7z_7 & 0 & y_3^{-1} \\ -y_4^{-1}y_2^{-1}y_6z_6 & -y_4^{-1}y_1^{-1}y_5z_5 & y_4^{-1} & 0 \end{bmatrix},$$

where $y_1, \dots, y_8 > 0$ and $z_5, z_6, z_7, z_8 \in \Omega$. Thus, the ray-pattern

$$A_0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ z_5 & z_6 & 0 & 1 \\ z_7 & z_8 & 1 & 0 \end{bmatrix}$$

is ICRN if and only if $z_8 = -z_5$ and $z_7 = -z_6$. Applying a suitable transformation

$$A_0 \;\mapsto\; (h \oplus g)A_0(h \oplus g)^{-1}, \qquad h, g \in \mathcal{G}_2,$$

we reduce A_0 to one of the forms indicated in (4.2) (if at least one of z_5, z_6, z_7, and z_8 is nonzero). □

We now state the main result of this paper describing all ICRN ray-patterns up to similarity with a similarity matrix in the group $\mathcal{G}$.

Theorem 4.2. *Let*

$$A = \begin{bmatrix} A_{1,1} & 0 & 0 & \dots & 0 \\ A_{2,1} & A_{2,2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{p,1} & A_{p,2} & A_{p,3} & \dots & A_{p,p} \end{bmatrix} \tag{4.3}$$

be an ICRN $n \times n$ ray-pattern, where $A_{1,1}, \dots, A_{p,p}$ are irreducible ICRN ray-patterns. Then there exists $g \in \mathcal{G}_n$ such that gAg^{-1} has a block lower triangular form

$$B := gAg^{-1} = [B_{i,j}]_{1 \le i,j \le p} = \left[\begin{array}{c|c} C_1 & 0 \\ \hline C_2 & C_3 \end{array}\right] \;\oplus\; C_0 \tag{4.4}$$

with the following properties:

(α) *C_0 is the direct sum of r identical matrices of the form* $\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$ *and of s identical matrices of the form* $\begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$;

(β)

$$\left[\begin{array}{c|c} C_1 & 0 \\ \hline 0 & C_3 \end{array}\right] = B_{1,1} \oplus \dots \oplus B_{q,q},$$

with $q = p - r - s$, where for $j = 1, \dots, q$:

$$B_{j,j} \in \left\{ [1], \quad [-1], \quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right\}; \tag{4.5}$$

(γ) *If the block* $\begin{bmatrix} B_{i,i} & 0 \\ B_{j,i} & B_{j,j} \end{bmatrix}$ *is such that $B_{j,i}$ is a submatrix of C_2, then*

$$\begin{bmatrix} B_{i,i} & 0 \\ B_{j,i} & B_{j,j} \end{bmatrix}$$

has one of the following forms:

$$\text{(a)}\ \begin{bmatrix} \pm 1 & 0 \\ 0 & \pm 1 \end{bmatrix}, \quad \text{(b)}\ \begin{bmatrix} \pm 1 & 0 \\ z & \mp 1 \end{bmatrix}, \quad \text{(c)}\ \begin{bmatrix} \pm 1 & 0 & 0 \\ z & 0 & 1 \\ \mp z & 1 & 0 \end{bmatrix},$$

$$\text{(d)}\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ z & \mp z & \pm 1 \end{bmatrix}, \qquad \text{(e)}\ \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ w & z & 0 & 1 \\ -z & -w & 1 & 0 \end{bmatrix}$$

for some $w, z \in \Omega$.

Conversely, every ray-pattern of the form (4.4) *with the properties* (α) - (γ) *is ICRN.*

The cases when one or more integers among q, r, s are zeros, with the obvious interpretation, are not excluded in Theorem 4.2. If $q = 1$, then the property (β) is interpreted in the sense that $B_{1,1}$ has one of the forms as in (4.5).

For convenience of reference, we define the following five types of block forms:

$$\text{(I)}\quad [1], \qquad \text{(II)}\quad [-1], \qquad \text{(III)}\quad \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\text{(IV)}\quad \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, \quad \text{(V)}\quad \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}. \tag{4.6}$$

The rather long proof of the theorem will be given in the next section.

It will be seen in the proof of Theorem 4.2 that when the proof is specialized to *real* ray-patterns A, the following real analogue of Theorem 4.2 is obtained. Let $\mathcal{G}_n^{(r)}$ be the group of $n \times n$ matrices of the form DQ, where Q is a permutation matrix, and D is a diagonal matrix with diagonal entries ± 1.

Theorem 4.3. *Let*

$$A = \begin{bmatrix} A_{1,1} & 0 & 0 & \dots & 0 \\ A_{2,1} & A_{2,2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{p,1} & A_{p,2} & A_{p,3} & \dots & A_{p,p} \end{bmatrix} \tag{4.7}$$

be an ICRN $n \times n$ *ray-pattern with entries* $0, \pm 1$, *where* $A_{1,1}, \dots, A_{p,p}$ *are irreducible ICRN ray-patterns. Then there exists* $g \in \mathcal{G}_n^{(r)}$ *such that* gAg^{-1} *has a block diagonal form* (4.4) *with the properties* (α), (β), *and* (γ), *where* $z, w \in \{0, 1, -1\}$ *in* (b) - (e). *Conversely, every real ray-pattern of the form as described is ICRN.*

Theorem 4.3 is essentially a reformulation of [1, Theorem 7.4.6].

5. Proof of Theorem 4.2

We prove first the direct statement of the theorem.

We use induction on p. For $p \leq 2$ the result is established in Proposition 3.1 and Lemma 4.1. Let now $p \geq 3$ and assume that Theorem 4.2 is proved for all smaller values of p.

Let A be given as in the Theorem. In what follows we formally put $A_{i,j} = 0$ if $i < j$. By Proposition 3.1 we may assume that each diagonal block $A_{j,j}$ has one of the forms (I) - (V).

Step 1. Suppose that one of the diagonal blocks, say A_{j_0,j_0} where $j_0 < p$, has the form (IV) or (V). Applying the induction hypothesis to $[A_{j,k}]_{j,k=1}^{p-1}$ we may assume that $A_{k,j_0} = 0$ for $k = j_0+1, \ldots, p-1$ and $A_{j_0,k} = 0$ for $k = 1, \ldots, j_0-1$. (if it happens that $A_{p,p}$ has one of the forms (IV) or (V), we apply the induction hypothesis to $[A_{j,k}]_{j,k=2}^{p}$ and argue similarly). Partition the matrix $[A_{j,k}]_{j,k=j_0}^{p}$ as follows:

$$[A_{j,k}]_{j,k=j_0}^{p} = \begin{bmatrix} A_{j_0,j_0} & 0 & 0 \\ 0 & C & 0 \\ A_{p,j_0} & D & A_{p,p} \end{bmatrix}, \tag{5.1}$$

where

$$C = [A_{j,k}]_{j,k=j_0+1}^{p-1}, \qquad D = [A_{p,j_0+1} \quad A_{p,j_0+2} \quad \ldots \quad A_{p,p-1}].$$

Since A is ICRN, it is easy to see that the matrix $[A_{j,k}]_{j,k=j_0}^{p}$ is also ICRN. For any positive matrix $X = [X_{u,v}]_{u,v=1}^{3}$ of appropriate size, partitioned conformably with (5.1), we have

$$\left(X \circ [A_{j,k}]_{j,k=j_0}^{p}\right)^{-1} = \begin{bmatrix} (X_{1,1} \circ A_{j_0,j_0})^{-1} & 0 & 0 \\ 0 & (X_{2,2} \circ C)^{-1} & 0 \\ Q & * & (X_{3,3} \circ A_{p,p})^{-1} \end{bmatrix},$$

where

$$Q = -(X_{3,3} \circ A_{p,p})^{-1}(X_{3,1} \circ A_{p,j_0})(X_{1,1} \circ A_{j_0,j_0})^{-1},$$

and

$$\mathrm{ray}\,(Q) = A_{p,j_0}.$$

On the other hand,

$$\left(\begin{bmatrix} X_{1,1} & X_{1,3} \\ X_{3,1} & X_{3,3} \end{bmatrix} \circ \begin{bmatrix} A_{j_0,j_0} & 0 \\ A_{p,j_0} & A_{p,p} \end{bmatrix}\right)^{-1} = \begin{bmatrix} (X_{1,1} \circ A_{j_0,j_0})^{-1} & 0 \\ Q & (X_{3,3} \circ A_{p,p})^{-1} \end{bmatrix},$$

and therefore the ray-pattern $\begin{bmatrix} A_{j_0,j_0} & 0 \\ A_{p,j_0} & A_{p,p} \end{bmatrix}$ is ICRN. By Lemma 4.1 $A_{p,j_0} = 0$. Permuting the j_0-th and p-th block rows and columns of A, we obtain a block diagonal matrix

$$\begin{bmatrix} * & 0 \\ 0 & A_{j_0,j_0} \end{bmatrix},$$

and an application of the induction hypothesis completes the proof in this case.

Step 2. Thus, we assume from now on in the proof that the blocks $A_{j,j}$ have forms (I), (II), (III), and therefore $q = p$ in our notation. Using the induction hypothesis we assume also that

$$\begin{bmatrix} A_{1,1} & 0 & 0 & \dots & 0 \\ A_{2,1} & A_{2,2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ A_{p-1,1} & A_{p-1,2} & A_{p-1,3} & \dots & A_{p-1,p-1} \end{bmatrix} = \begin{bmatrix} B_{1,1} & 0 & 0 & \dots & 0 \\ B_{2,1} & B_{2,2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ B_{p-1,1} & B_{p-1,2} & B_{p-1,3} & \dots & B_{p-1,p-1} \end{bmatrix},$$

where the $B_{j,k}$'s have the properties described in the theorem.

We prove the property (β) first. As an intermediate step, the following statement will be proved:

(A) *For every index i, $2 \le i \le p-1$, either the blocks $A_{i,1}, \dots, A_{i,i-1}$ are all zeros, or the blocks $A_{i+1,1}, \dots, A_{p,i}$ are all zeros, or both.*

Arguing by contradiction, suppose there exist an index i_0, $2 \le i_0 \le p-1$, such that not all blocks $A_{i_0,1}, \dots, A_{i_0,i_0-1}$ are zeros and not all blocks $A_{i_0+1,i_0}, \dots, A_{p,i_0}$ are zeros. We select the smallest i_0 with these properties. Because of the induction hypothesis made above,

$$A_{i_0+1,i_0} = 0, \dots, A_{p-1,i_0} = 0, A_{p,i_0} \neq 0. \tag{5.2}$$

The matrix $[A_{j,k}]_{j,k=i_0}^{p}$ can be partitioned as follows:

$$[A_{j,k}]_{j,k=i_0}^{p} = \begin{bmatrix} A_{i_0,i_0} & 0 & 0 \\ 0 & C & 0 \\ A_{p,i_0} & D & A_{p,p} \end{bmatrix}, \tag{5.3}$$

analogously to (5.1), and as in the proof of Step 1, we conclude that the submatrix

$$\begin{bmatrix} A_{i_0,i_0} & 0 \\ A_{p,i_0} & A_{p,p} \end{bmatrix}$$

is ICRN. Thus, by Lemma 4.1 the cases when $A_{i_0,i_0} = A_{p,p} = \pm 1$ cannot occur. Denoting by n_j ($n_j \in \{1,2\}$) the size of the block $A_{j,j}$, and applying a similarity transformation

$$A \ \mapsto \ (h \oplus g)A(h \oplus g)^{-1},$$

where $h \in \mathcal{G}_{n_{i_0}}$, $g \in \mathcal{G}_{n_p}$, we may assume, in view of the same Lemma 4.1, in addition to the assumptions already made, that the block A_{p,i_0} has the following structure:

(α') if $A_{i_0,i_0} = -A_{p,p} = \pm 1$, then $A_{p,i_0} = 1$;

(β') if $A_{i_0,i_0} = \pm 1$ and $A_{p,p} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $A_{p,i_0} = \begin{bmatrix} 1 \\ \mp 1 \end{bmatrix}$;

(γ') if $A_{i_0,i_0} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $A_{p,p} = \mp 1$, then $A_{p,i_0} = [1 \quad \pm 1]$;

(δ') if $A_{i_0,i_0} = A_{p,p} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, then $A_{p,i_0} = \begin{bmatrix} 1 & z \\ -z & -1 \end{bmatrix}$ for some $z \in \Omega$.

Let $X = [X_{i,j}]_{i,j=1}^{p}$ be a positive matrix partitioned conformably with the partition of A. Partition $(X \circ A)^{-1}$, again conformably with that of A:

$$(X \circ A)^{-1} = \begin{bmatrix} Q_{1,1} & 0 & 0 & \dots & 0 \\ Q_{2,1} & Q_{2,2} & 0 & \dots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ Q_{p,1} & Q_{p,2} & Q_{p,3} & \dots & Q_{p,p} \end{bmatrix}.$$

We have

$$\begin{aligned} Q_{p,1} &= -(X_{p,p} \circ A_{p,p})^{-1}(X_{p,1} \circ A_{p,1})(X_{1,1} \circ A_{1,1})^{-1} \\ &+ (X_{p,p} \circ A_{p,p})^{-1} \left[X_{p,2} \circ A_{p,2} \quad \dots \quad X_{p,p-1} \circ A_{p,p-1}\right] \\ &\times \left[[X_{i,j} \circ A_{i,j}]_{i,j=2}^{p-1}\right]^{-1} \begin{bmatrix} X_{2,1} \circ A_{2,1} \\ X_{3,1} \circ A_{3,1} \\ \vdots \\ X_{p-1,1} \circ A_{p-1,1} \end{bmatrix} (X_{1,1} \circ A_{1,1})^{-1}. \end{aligned}$$

Since $A_{2,1} = \ldots = A_{i_0-1,1} = 0$, the above expression for $Q_{p,1}$ can be rewritten in the form

$$\begin{aligned} Q_{p,1} &= -(X_{p,p} \circ A_{p,p})^{-1}(X_{p,1} \circ A_{p,1})(X_{1,1} \circ A_{1,1})^{-1} \\ &+ (X_{p,p} \circ A_{p,p})^{-1} \left[X_{p,i_0} \circ A_{p,i_0} \quad \dots \quad X_{p,p-1} \circ A_{p,p-1}\right] \\ &\times \left[[X_{i,j} \circ A_{i,j}]_{i,j=i_0}^{p-1}\right]^{-1} \begin{bmatrix} X_{i_0,1} \circ A_{i_0,1} \\ X_{i_0+1,1} \circ A_{i_0+1,1} \\ \vdots \\ X_{p-1,1} \circ A_{p-1,1} \end{bmatrix} (X_{1,1} \circ A_{1,1})^{-1}. \end{aligned} \tag{5.4}$$

Because of (5.2),

$$\begin{aligned} Q_{p,1} &= (X_{p,p} \circ A_{p,p})^{-1}(X_{p,i_0} \circ A_{p,i_0})(X_{i_0,i_0} \circ A_{i_0,i_0})^{-1}(X_{i_0,1} \circ A_{i_0,1})(X_{1,1} \circ A_{1,1})^{-1} \\ &+ \{\text{terms that are independent of } X_{p,i_0} \text{ and of } X_{i_0,1}\}. \end{aligned} \tag{5.5}$$

We let

$$W = (X_{p,p} \circ A_{p,p})^{-1}(X_{p,i_0} \circ A_{p,i_0})(X_{i_0,i_0} \circ A_{i_0,i_0})^{-1}(X_{i_0,1} \circ A_{i_0,1})(X_{1,1} \circ A_{1,1})^{-1}.$$

Step 3. Consider the case when all entries of W are nonzero, for some choice of positive matrices

$$X_{p,p}, \quad X_{p,i_0}, \quad X_{i_0,i_0}, \quad X_{i_0,1}, \quad \text{and} \quad X_{1,1}. \tag{5.6}$$

In view of the conditions (α') - (γ') and (a) - (e) (the latter conditions, with $B_{i,j}$ replaced by $A_{i,j}$ are satisfied by the blocks of the matrix $[A_{i,j}]_{i,j=1}^{p-1}$), this case *does not occur* only if

$$A_{1,1} = A_{i_0,i_0} = A_{p,p} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \tag{5.7}$$

and we will consider separately the case when (5.7) holds true. Let the matrices (5.6) be fixed so that all entries of W are nonzero. By keeping all other diagonal blocks of X fixed, and choosing all other nondiagonal blocks of X sufficiently small, we have by virtue of (5.5) that

$$\|Q_{p,1} - W\| < \epsilon, \tag{5.8}$$

for any prescribed $\epsilon > 0$. On the other hand, since A is ICRN, we have that ray $(Q_{p,1}) = A_{p,1}$ is independent on X, and since all entries of W are nonzero, (5.5) and (5.8) give

$$\text{ray}\,(W) = A_{p,1} \qquad \text{(provided } W \text{ has no zero entries).} \tag{5.9}$$

(Use here the fact that the ray-pattern projection P is continuous on the set of matrices with no zero entries.) In particular,

$$\text{the matrix } A_{p,1} \text{ has no zero entries,} \tag{5.10}$$

and since ray $(Q_{p,1}) = A_{p,1}$, it follows that $Q_{p,1}$ has no zero entries either. Returning to formulas (5.4) and (5.5), write

$$\begin{aligned} Q_{p,1} &= -(X_{p,p} \circ A_{p,p})^{-1}(X_{p,1} \circ A_{p,1})(X_{1,1} \circ A_{1,1})^{-1} + W \\ &+ \{\text{terms that are independent of } X_{p,i_0}, \text{ of } X_{i_0,1} \text{ and of } X_{p,1}\}. \end{aligned} \tag{5.11}$$

Keeping in this formula $X_{p,1}$, $X_{1,1}$, and $X_{p,p}$ fixed, keeping fixed all other diagonal blocks of X, and letting all other nondiagonal blocks of X tend to zero, we obtain

$$\|Q_{p,1} - \left(-(X_{p,p} \circ A_{p,p})^{-1}(X_{p,1} \circ A_{p,1})(X_{1,1} \circ A_{1,1})^{-1}\right)\| < \epsilon,$$

for any prescribed $\epsilon > 0$. Note that since $A_{p,1}$ has no zero entries, and since $A_{1,1}$ and $A_{p,p}$ have the forms (I), (II), or (III), it is easy to see that the matrix

$$U := -(X_{p,p} \circ A_{p,p})^{-1}(X_{p,1} \circ A_{p,1})(X_{1,1} \circ A_{1,1})^{-1}$$

has no zero entries. Using again the continuity of the ray-pattern projection on the set of matrices with no zero entries, with U playing the role of W in the above argument, it follows (letting ϵ tend to zero) that

$$\text{ray}\,(U) = \text{ray}\,(Q_{p,1}) = A_{p,1}. \tag{5.12}$$

As a result we obtain that the matrix

$$\begin{bmatrix} A_{1,1} & 0 \\ A_{p,1} & A_{p,p} \end{bmatrix} \tag{5.13}$$

is ICRN.

We now consider several situations that may occur.

(S1) $A_{1,1} = \pm 1$. Then $A_{i_0,i_0} \neq \pm 1$ (or else the block $A_{i_0,1}$ would have been zero, by (a) - (e)), and if $A_{i_0,i_0} = \mp 1$, then $A_{p,p} \neq \mp 1$ (for a similar reason, in view of (α') - (δ')). Also, by (5.10) and (5.13), $A_{p,p} \neq \pm 1$. Thus, we have the following situations:

$$\text{(S11)} \quad A_{1,1} = \pm 1, \quad A_{i_0,i_0} = \mp 1, \quad A_{p,p} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix};$$

$$\text{(S12)}\quad A_{1,1}=\pm 1,\quad A_{i_0,i_0}=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad A_{p,p}=\mp 1;$$

$$\text{(S13)}\quad A_{1,1}=\pm 1,\quad A_{i_0,i_0}=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad A_{p,p}=\begin{bmatrix}0&1\\1&0\end{bmatrix}.$$

(S2) $A_{1,1}=\begin{bmatrix}0&1\\1&0\end{bmatrix}$. Then $A_{i_0,i_0}=A_{p,p}=\pm 1$ is impossible, and (5.7) is excluded (so far), so we have the following situations:

$$\text{(S21)}\quad A_{1,1}=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad A_{i_0,i_0}=\pm 1,\quad A_{p,p}=\mp 1;$$

$$\text{(S22)}\quad A_{1,1}=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad A_{i_0,i_0}=\pm 1,\quad A_{p,p}=\begin{bmatrix}0&1\\1&0\end{bmatrix};$$

$$\text{(S23)}\quad A_{1,1}=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad A_{i_0,i_0}=\begin{bmatrix}0&1\\1&0\end{bmatrix},\quad A_{p,p}=\pm 1.$$

We now consider each of the six situations (S11) - (S13), (S21) - (S23) separately. In these considerations, it will be assumed that the matrices $X_{1,1}$, X_{i_0,i_0} and $X_{p,p}$ are matrices of all 1's.

Assume (S11), and let

$$\begin{bmatrix}X_{1,1}&*&*\\X_{i_0,1}&X_{i_0,i_0}&*\\X_{p,1}&X_{p,i_0}&X_{p,p}\end{bmatrix}=\begin{bmatrix}1&*&*&\\x_2&1&*&\\x_4&x_6&1&1\\x_5&x_7&1&1\end{bmatrix},\quad x_2,\ldots,x_7>0.$$

By virtue of (a) - (e), (α') - (δ'), (5.10), and (5.13), we have

$$A_{p,i_0}=\begin{bmatrix}1\\\pm 1\end{bmatrix},\quad A_{i_0,1}=[1],\quad A_{p,1}=\begin{bmatrix}z\\\mp z\end{bmatrix},\quad |z|=1.$$

A computation shows that

$$W=\begin{bmatrix}\mp x_7x_2\\-x_6x_2\end{bmatrix},\quad U=\begin{bmatrix}x_4z\\\mp x_4z\end{bmatrix}.$$

We have

$$\operatorname{ray}(W)\neq\operatorname{ray}(U),$$

a contradiction with (5.9) and (5.12).

Assume (S12), and let

$$\begin{bmatrix}X_{1,1}&*&*\\X_{i_0,1}&X_{i_0,i_0}&*\\X_{p,1}&X_{p,i_0}&X_{p,p}\end{bmatrix}=\begin{bmatrix}1&*&&*\\x_2&1&1&*\\x_3&1&1&\\x_8&x_9&x_{10}&1\end{bmatrix},\quad x_2,\ldots,x_{10}>0.$$

Also,

$$A_{p,i_0}=[1\quad\pm 1],\quad A_{i_0,1}=\begin{bmatrix}1\\\mp 1\end{bmatrix},\quad A_{p,1}=[z],\quad |z|=1.$$

A computation shows that $W = \pm x_{10}x_2 \mp x_9 x_8$. Thus, ray (W) is not constant, a contradiction with (5.9).

Assume (S13), and let

$$\begin{bmatrix} X_{1,1} & * & * \\ X_{i_0,1} & X_{i_0,i_0} & * \\ X_{p,1} & X_{p,i_0} & X_{p,p} \end{bmatrix} = \begin{bmatrix} 1 & * & * \\ x_2 & \begin{matrix} 1 & 1 \end{matrix} & * \\ x_3 & \begin{matrix} 1 & 1 \end{matrix} & \\ x_8 & \begin{matrix} x_{10} & x_{11} \end{matrix} & \begin{matrix} 1 & 1 \end{matrix} \\ x_9 & \begin{matrix} x_{12} & x_{13} \end{matrix} & \begin{matrix} 1 & 1 \end{matrix} \end{bmatrix}, \quad x_2, \ldots, x_{13} > 0.$$

Also, we have

$$A_{p,i_0} = \begin{bmatrix} 1 & u \\ -u & -1 \end{bmatrix} \quad \text{for some} \quad |u| = 1, \quad \text{or} \quad A_{p,i_0} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix},$$

$$A_{i_0,1} = \begin{bmatrix} 1 \\ \mp 1 \end{bmatrix}, \quad A_{p,1} = \begin{bmatrix} z \\ \mp z \end{bmatrix} \quad \text{for some} \quad |z| = 1.$$

A computation shows that

$$W = \begin{bmatrix} x_{12}ux_3 \mp x_{13}x_2 \\ -x_{10}x_3 \pm x_{11}ux_2x_1^{-1} \end{bmatrix} \quad \text{or} \quad W = \begin{bmatrix} x_{12}x_3 \\ \pm x_{11}x_2 \end{bmatrix}, \tag{5.14}$$

depending on the form of A_{p,i_0}, and

$$U = \begin{bmatrix} x_9 z \\ \mp x_8 z \end{bmatrix}.$$

If W has the second form in (5.14), then clearly ray $(W) \neq$ ray (U), a contradiction. If W has the first form in (5.14), then ray (W) is not constant (which happens if $u \neq -1$ in the case of upper signs, or if $u \neq 1$ in the case of lower signs), or ray (W) is constant and ray $(W) \neq$ ray (U) (which happens if the signs are lower and $u = 1$), or ray (W) is constant and ray $(W) =$ ray (U) (which happens if the signs are upper and $u = -1$), but then ray $(W) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$, a contradiction in all cases; in the latter case, a contradiction is obtained with ray $(W) = A_{p,1}$.

The cases (S21) and (S23) can be reduced to (S11) and (S13), respectively, by taking transposed matrices, and then permuting rows and columns appropriately to get lower triangular forms. Thus, we obtain a contradiction in the cases (S21) and (S23).

Assume (S22), and let

$$\begin{bmatrix} X_{1,1} & * & * \\ X_{i_0,1} & X_{i_0,i_0} & * \\ X_{p,1} & X_{p,i_0} & X_{p,p} \end{bmatrix} = \begin{bmatrix} \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} & * & * \\ \begin{matrix} x_5 & x_6 \end{matrix} & 1 & * \\ \begin{matrix} x_8 & x_9 \\ x_{10} & x_{11} \end{matrix} & \begin{matrix} x_{12} \\ x_{13} \end{matrix} & \begin{matrix} 1 & 1 \\ 1 & 1 \end{matrix} \end{bmatrix}, \quad x_5, \ldots, x_{13} > 0.$$

Here,

$$A_{i_0,1} = [1 \quad \mp 1], \quad A_{p,i_0} = \begin{bmatrix} 1 \\ \mp 1 \end{bmatrix},$$

and the property that the matrix $\begin{bmatrix} A_{1,1} & 0 \\ A_{p,1} & A_{p,p} \end{bmatrix}$ is ICRN together with $A_{p,1}$ not having zero entries imply that

$$A_{p,1} = \begin{bmatrix} u & v \\ -v & -u \end{bmatrix}$$

for some u, v with $|u| = |v| = 1$. A computation shows that

$$W = \begin{bmatrix} \pm x_{13}x_6 & -x_{13}x_5 \\ -x_{12}x_6 & \pm x_{12}x_5 \end{bmatrix} \quad \text{and} \quad U = \begin{bmatrix} x_{11}u & x_{10}v \\ -x_9 v & -x_8 u \end{bmatrix}.$$

Clearly, $\operatorname{ray}(W) \neq \operatorname{ray}(U)$, a contradiction again.

Step 4. Consider the so far excluded case when (5.7) holds true. Applying a transformation

$$A \ \mapsto \ (h \oplus I)A(h \oplus I)^{-1}, \quad h \in \mathcal{G}_2,$$

the block $A_{i_0,1}$ may be transformed to the form

$$\begin{bmatrix} 1 & z \\ -z & -1 \end{bmatrix}, \quad z \in \Omega. \tag{5.15}$$

Thus we assume that $A_{i_0,1}$ and A_{p,i_0} have the form (5.15).

Assume first that at least one of $A_{i_0,1}$ and A_{p,i_0} has the form (5.15) with $|z| = 1$. Then for some choice of the matrices $X_{i,j}$, $i, j \in \{1, i_0, p\}$, $i \neq j$, still keeping the matrices $X_{i,i}$, $i \in \{1, i_0, p\}$ all 1's, the matrix W has no zero entries, and we may repeat the arguments of Step 3. Thus, the properties obtained in Step 3 are valid. In particular the matrix

$$\begin{bmatrix} A_{1,1} & 0 \\ A_{p,1} & A_{p,p} \end{bmatrix}$$

is ICRN, and therefore (also because $A_{p,1}$ has no zero entries) we have

$$A_{p,1} = \begin{bmatrix} u & v \\ -v & -u \end{bmatrix}, \quad |u| = |v| = 1.$$

Let

$$A_{i_0,1} = \begin{bmatrix} 1 & z \\ -z & -1 \end{bmatrix}, \quad z \in \Omega, \quad A_{p,i_0} = \begin{bmatrix} 1 & w \\ -w & -1 \end{bmatrix}, \quad w \in \Omega,$$

where not both z and w are zero. As in Step 3, we compute W and U and obtain a contradiction with one of the properties of W and U. We let

$$\begin{bmatrix} X_{1,1} & * & * \\ X_{i_0,1} & X_{i_0,i_0} & * \\ X_{p,1} & X_{p,i_0} & X_{p,p} \end{bmatrix} = \begin{bmatrix} 1 & 1 & & & & \\ 1 & 1 & & * & & * \\ x_5 & x_6 & 1 & 1 & & \\ x_7 & x_8 & 1 & 1 & & * \\ x_{13} & x_{14} & x_{17} & x_{18} & 1 & 1 \\ x_{15} & x_{16} & x_{19} & x_{20} & 1 & 1 \end{bmatrix}, \quad x_5, \ldots, x_{20} > 0.$$

A computation shows that

$$W = \begin{bmatrix} -x_{20}x_6 z + x_{19} w x_8 & -x_{20}x_5 + x_{19} w x_7 z \\ x_{18} w x_6 z - x_{17}x_8 & x_{18} w x_5 - x_{17} x_7 z \end{bmatrix}, \quad U = \begin{bmatrix} x_{16}u & x_{15}v \\ -x_{14}v & -x_{13}u \end{bmatrix}. \tag{5.16}$$

If ray (W) is not constant (i.e., independent of x_j as long as W has no zero entries), we obtain a contradiction with (5.9). If ray (W) is constant, then its off diagonal entries must be both negative, whereas the off diagonal entries of U are negative multiples of each other (or are equal to zero if $v = 0$). Thus, ray $(W) \neq$ ray (U), a contradiction again, with (5.9) and (5.12).

Step 5. We take up the remaining case when (5.7) holds true, and

$$A_{i_0,1} = A_{p,i_0} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Partition the positive matrix $X = [X_{i,j}]_{i,j=1}^p$ and $(X \circ A)^{-1} = [Q_{i,j}]_{i,j=1}^p$ conformably with (4.3), and fix $X_{1,1}$, X_{i_0,i_0}, and $X_{p,p}$ to be the matrices of all 1's. Then $Q_{p,1}$ takes the form (cf. (5.11)):

$$\begin{aligned} Q_{p,1} &= -\left(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (X_{p,1} \circ A_{p,1}) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right) + W \\ &+ \{\text{terms that are independent of } X_{p,i_0}, \text{ of } X_{i_0,1} \text{ and of } X_{p,1}\}, \end{aligned} \tag{5.17}$$

where

$$W = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (X_{p,i_0} \circ A_{p,i_0}) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} (X_{i_0,1} \circ A_{i_0,1}) \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Let

$$X_{i_0,1} = \begin{bmatrix} x_1 & x_2 \\ x_3 & x_4 \end{bmatrix}, \quad X_{p,1} = \begin{bmatrix} x_5 & x_6 \\ x_7 & x_8 \end{bmatrix}, \quad X_{p,i_0} = \begin{bmatrix} x_9 & x_{10} \\ x_{11} & x_{12} \end{bmatrix}, \quad A_{p,1} = \begin{bmatrix} z_1 & z_2 \\ z_3 & z_4 \end{bmatrix}.$$

A computation shows that the matrix $Q_{p,1}$ has the form

$$Q_{p,1} = \begin{bmatrix} -x_8 z_4 & -x_7 z_3 - x_{12} x_1 \\ -x_6 z_2 - x_9 x_4 & -x_5 z_1 \end{bmatrix} + \{ \dots \}, \tag{5.18}$$

where the ellipse stands for terms that are independent of $x_1, \dots, x_{12}$.

Let us analyze (5.18). Regardless of the value of z_2, the $(2,1)$ entry of

$$\begin{bmatrix} -x_8 z_4 & -x_7 z_3 - x_{12} x_1 \\ -x_6 z_2 - x_9 x_4 & -x_5 z_1 \end{bmatrix} + \{ \dots \}$$

is nonzero for some x_6, x_9, x_4. Fix these values, and (for a given $\epsilon > 0$) select the blocks of X other than $X_{i,j}$, $i,j \in \{1, i_0, p\}$, so that

$$\| Q_{p,1} - \begin{bmatrix} -x_8 z_4 - x_{12} x_2 & -x_7 z_3 \\ -x_6 z_2 & -x_5 z_1 - x_9 x_3 \end{bmatrix} \| < \epsilon. \tag{5.19}$$

Since ray $(Q_{p,1}) = A_{p,1}$, passing to the limit in (5.19) as $\epsilon \longrightarrow 0$ we obtain that

$$-x_6 z_2 - x_9 x_4 = c z_3$$

for some positive c, as long as $-x_6 z_2 - x_9 x_4 \neq 0$. Thus,

$$z_2 = -z_3 = 1. \tag{5.20}$$

Applying a similar argument to the $(1,2)$ entry of $Q_{p,1}$ we obtain $z_3 = -z_2 = 1$, a contradiction with (5.20).

This completes the proof of the statement (A) for the matrix A (after a similarity $A \mapsto gAg^{-1}$, for some $g \in \mathcal{G}$).

Property (β) is now deduced without difficulties. Still using the induction hypothesis, assume that

$$A = [A_{j,k}]_{j,k=1}^{p-1} = [B_{j,k}]_{j,k=1}^{p-1},$$

where the $B_{j,k}$'s have the properties as in Theorem 4.2, and let t be such that each of $[A_{i,j}]_{i,j=1}^{t}$ and $[A_{i,j}]_{i,j=t+1}^{p-1}$ is a direct sum of matrices of forms (I), (II), and (III). If for some index $s \geq t+1$ it happens that $A_{p,s} \neq 0$, then by the already proven part we have

$$A_{1,s} = 0, \dots, A_{s-1,s} = 0.$$

If $s = t+1$, then the block $[A_{i,j}]_{i,j=1}^{t+1}$ is a direct sum of matrices of forms (I), (II), and (III). If $s > t+1$, then we interchange in A the s-th and $(t+1)$-th block rows, and the s-th and $(t+1)$-th block columns, resulting in a matrix whose $(t+1) \times (t+1)$ block upper left corner is a direct sum of matrices of forms (I), (II), and (III). Continuing this process we eventually obtain a matrix B with property (β).

Property (γ) will follow by elementary considerations from Property (β) (cf. the proof of Lemma 4.1) once we show that all ray-patterns

$$\begin{bmatrix} B_{i,i} & 0 \\ B_{j,i} & B_{j,j} \end{bmatrix} \qquad 1 \leq i \leq t < j \leq p,$$

are ICRN. To this end, let X be a positive matrix of the same size as B, and partition:

$$(X \circ B)^{-1} = [Y_{k,m}]_{k,m=1}^{p}.$$

It is clear from property (β) that for fixed i, j $(1 \leq i \leq t < j \leq p)$ we have

$$Y_{j,i} = -(X_{j,j} \circ B_{j,j})^{-1}(X_{j,i} \circ B_{j,i})(X_{i,i} \circ B_{i,i})^{-1},$$

and by the ICRN property of B it follows that

$$\operatorname{ray}(Y_{j,i}) = B_{j,i}. \tag{5.21}$$

On the other hand, $Y_{j,i}$ is also the off-diagonal block of the matrix

$$\left(\begin{bmatrix} X_{i,i} & X_{i,j} \\ X_{j,i} & X_{j,j} \end{bmatrix} \circ \begin{bmatrix} B_{i,i} & 0 \\ B_{j,i} & B_{j,j} \end{bmatrix} \right)^{-1},$$

and the ICRN property of $\begin{bmatrix} B_{i,i} & 0 \\ B_{j,i} & B_{j,j} \end{bmatrix}$ follows from (5.21).

Finally, the converse part of Theorem 4.2 is verified in a straightforward way, taking into account that the ray-pattern B given by (4.4) with the property (β) is ICRN if and only if each of the 2×2 blocks $\begin{bmatrix} B_{i,i} & 0 \\ B_{j,i} & B_{j,j} \end{bmatrix}$, where $1 \leq i \leq t < j \leq p$, is ICRN (see the preceding paragraph).

6. Inertias and Jordan forms of matrices with ICRN ray-patterns

We start with inertia considerations. Let $Y \in M_n(\mathbb{C})$, and assume that Y has no eigenvalues on the imaginary axis. We define the inertia $\mathrm{In}\, Y = \{i_-(Y), i_+(Y)\}$, where $i_-(Y)$ (resp., $i_+(Y)$) is the number of eigenvalues of Y (counted with multiplicities) in the open left (resp., right) halfplane. Thus, $i_-(Y) + i_+(Y) = n$.

Lemma 6.1. *Let A be an ICRN ray-pattern, and let $\mathrm{Cone}\,(A)$ be the cone generated by A. Then the spectrum of every $Y \in \mathrm{Cone}\,(A)$ does not intersect the imaginary axis. Moreover, there exist two nonnegative integers $i_\pm(A)$, depending on A only, that sum up to n such that*

$$\mathrm{In}\, Y = \{i_-(A), i_+(A)\}$$

for every $Y \in \mathrm{Cone}\,(A)$.

Proof. Let $Y \in \mathrm{Cone}\,(A)$, and arguing by contradiction assume $Yx = i\lambda x$ for some nonzero x and $\lambda \in \mathbb{R} \setminus \{0\}$. Then

$$Y^{-1}x = -i\frac{1}{\lambda}x.$$

Now

$$\left(\frac{1}{|\lambda|}Y + |\lambda|Y^{-1}\right)x = 0,$$

which is impossible because

$$\frac{1}{|\lambda|}Y + |\lambda|Y^{-1} \in \mathrm{Cone}\,(A),$$

and therefore $\frac{1}{|\lambda|}Y + |\lambda|Y^{-1}$ is invertible in view of the inverse closeness property of A. The second statement of the lemma follows easily from the first, using a standard argument that involves arcwise connectedness of $\mathrm{Cone}\,(A)$ and continuity of eigenvalues of Y. $\square$

We define the *inertia* of an ICRN ray-pattern A as the numbers $i_\pm(A)$ introduced in Lemma 6.1. The inertia of ICRN ray-patterns are described as follows.

Theorem 6.2. *If A is an $n \times n$ ICRN ray-pattern, then $\mathrm{trace}\, A$ is an integer of the same parity as n, $|\mathrm{trace}\, A| \leq n$, and*

$$i_\pm(A) = \frac{n \pm \mathrm{trace}\, A}{2}.$$

In particular, all matrices in $\mathrm{Cone}\,(A)$ are stable (i.e., all eigenvalues have negative real parts) if and only if $A = -I$.

Proof. In view of Theorem 4.2, we need only to prove the result for each of the five forms (3.1). For the first four forms this is immediate, for the form (V) it follows from Lemma 6.3 below (which will be also used in the proof of the next theorem). □

Next, we turn to the eigenvalues and Jordan forms of matrices in Cone(A). We first establish two lemmas.

Lemma 6.3. *Let*

$$Y = \begin{bmatrix} 0 & 0 & a & b \\ 0 & 0 & c & -d \\ e & f & 0 & 0 \\ g & -h & 0 & 0 \end{bmatrix}, \qquad a,b,c,d,e,f,g,h > 0.$$

Then:

(1) *Y has no eigenvalues with zero real part;*
(2) *if λ is an eigenvalue of Y, then so is $-\lambda$, and the algebraic multiplicities of $-\lambda$ and of λ are the same;*
(3) *if $\lambda = \mu + i\nu \in \mathbb{C}$, $\mu, \nu \in \mathbb{R}$, is an eigenvalue of Y, then $|\nu| < |\mu|$.*

Conversely, if $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ is a quadruple of not necessarily distinct complex numbers, which is closed under negation and complex conjugation, and satisfies

$$|\operatorname{Im}\lambda_j| < |\operatorname{Re}\lambda_j|, \qquad j = 1,2,3,4,$$

then there exist $a,b,c,d,e,f,g,h > 0$ such that

$$\sigma(Y) = \{\lambda_1, \lambda_2, \lambda_3, \lambda_4\}. \tag{6.1}$$

Proof. (1) follows from Lemma 6.1, because

$$A := \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 \\ 1 & 1 & 0 & 0 \\ 1 & -1 & 0 & 0 \end{bmatrix}$$

is an ICRN. The characteristic polynomial of Y is computed to be

$$\lambda^4 + \lambda^2(-dh - cf - ae - bg) + (aedh + adfg + behc + bcfg).$$

Thus, (2) follows. Moreover, $\operatorname{Re}(\lambda^2) > 0$ for every eigenvalue λ of Y. Property (3) now follows.

For the converse statement observe that by letting $f = d = h = c$ and $e = b = g = a$, the characteristic polynomial of Y takes the form

$$\lambda^4 + \lambda^2(-2c^2 - 2a^2) + 4a^2c^2 = (\lambda^2 - 2a^2)(\lambda^2 - c^2),$$

and therefore for every quadruple of nonzero real numbers $(\lambda_1, \lambda_2, \lambda_3, \lambda_4)$ which is closed under negation there exist $a,b,c,d,e,f,g,h > 0$ such that (6.1) holds. Next, fix $\mu > 0$, and let

$$d = h = c = f = a = e = \sqrt{\mu}, \qquad g = \frac{\mu}{b}.$$

Then the characteristic polynomial of Y is

$$\lambda^4 + \lambda^2(-4\mu) + (2\mu^2 + \mu\sqrt{\mu}(b + \frac{\mu}{b})),$$

and therefore for eigenvalues λ_0 of Y we have

$$\lambda_0^2 = 2\mu \pm i\sqrt{\mu\sqrt{\mu}(b + \frac{\mu}{b}) - 2\mu^2}.$$

Clearly, by a suitable choice of $\mu > 0$ and $b > 0$, λ_0^2 can be made equal to any nonreal complex number $z = \alpha + i\beta$ with positive real part α. Indeed, let $\mu = \alpha/2$ and b a positive solution of the quadratic equation

$$b^2 - 2\sqrt{\mu}b + \mu - \frac{\beta^2}{\mu\sqrt{\mu}} = 0.$$

Lemma 6.3 is proved. □

Lemma 6.4. *(a) Suppose $B_1 = [B_{i,j}]_{i,j=1}^q$ is a block lower triangular matrix in the form (4.4) satisfying the conditions (1) – (2) of Theorem 4.2. Let X be a matrix with positive entries such that $\lambda \in \mathbb{R} \setminus \{0\}$ is an eigenvalue of $X \circ B$. Then $X \circ B$ is similar to a matrix of the form $T \oplus U$, where*

$$T = \begin{bmatrix} \lambda I_\alpha & 0 & 0 & 0 \\ 0 & \lambda I_x & 0 & 0 \\ 0 & R_1 & \lambda I_\gamma & 0 \\ R_2 & R_3 & 0 & \lambda I_y \end{bmatrix} \oplus \begin{bmatrix} -\lambda I_\beta & 0 & 0 & 0 \\ 0 & -\lambda I_x & 0 & 0 \\ 0 & S_1 & -\lambda I_\delta & 0 \\ S_2 & S_3 & 0 & -\lambda I_y \end{bmatrix}, \tag{6.2}$$

and none of $\pm\lambda$ is an eigenvalue of U.

(b) If T has the form (6.2) with $\lambda \neq 0$, for some matrices R_j, S_j ($j = 1, 2, 3$) of suitable sizes, then the Jordan form of T consists of blocks of the forms:

$$[\lambda], \quad [-\lambda], \quad \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \quad \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}$$

with multiplicities r_1, s_1, r_2, s_2, respectively, such that

$$r_1 + 2r_2 \geq s_2 \qquad \text{and} \qquad s_1 + 2s_2 \geq r_2. \tag{6.3}$$

Proof. Note that $X \circ B$ has the form

$$\begin{bmatrix} C_{1,1} & 0 \\ C_{2,1} & C_{2,2} \end{bmatrix}, \tag{6.4}$$

such that each of $C_{1,1}$ and $C_{2,2}$ is a direct sum of 1×1 and 2×2 matrices. Applying a block permutation similarity, we can assume that

$$C_{11} = U_1 \oplus \lambda I_\alpha \oplus -\lambda I_\beta \oplus X' \qquad \text{and} \qquad C_{22} = \lambda I_\gamma \oplus -\lambda I_\delta \oplus Y' \oplus U_2,$$

such that none of $\pm\lambda$ is an eigenvalue of U_1 or U_2, and each of X' and Y' is a direct sum of 2×2 real matrices with zero diagonal and eigenvalues $\pm\lambda$. Thus, X' and Y' are similar to the matrices $\lambda(I_x \oplus -I_x)$ and $\lambda(I_y \oplus -I_y)$, respectively. We may apply a similarity transformation to $[C_{i,j}]_{i,j=1}^2$ so that X' and Y' are changed to $\lambda(I_x \oplus -I_x)$ and $\lambda(I_y \oplus -I_y)$, respectively.

We now write (6.4), after a transformation as indicated above, as the block lower triangular matrix $V = [V_{i,j}]_{i,j=1}^{10}$ such that

$$[V_{i,j}]_{i,j=1}^{5} = U_1 \oplus \lambda I_\alpha \oplus -\lambda I_\beta \oplus \lambda I_x \oplus -\lambda I_x$$

and

$$[V_{i,j}]_{i,j=6}^{10} = \lambda I_\gamma \oplus -\lambda I_\delta \oplus \lambda I_y \oplus -\lambda I_y \oplus U_2.$$

By property (β) of Theorem 4.2, we see that $V_{6,2}$ and $V_{7,3}$ are zero blocks.

Next, we show that one can apply a sequence of block permutation similarity transformations to the matrix $[V_{i,j}]_{i,j=1}^{10}$ that convert all $V_{i,j}$ to zero except for

$$V_{6,4}, V_{7,5}, V_{8,2}, V_{8,4}, V_{9,3}, V_{9,5}, V_{10,1},$$

and keep the blocks $V_{6,2}$ and $V_{7,3}$ zeros. Then by a block similarity transformation, the resulting matrix will be similar to $T \oplus U$ with

$$U = \begin{bmatrix} U_1 & 0 \\ V_{10,1} & U_2 \end{bmatrix},$$

and with T given by (6.2). In the following, all block matrices having the same size as V are partitioned according to the 10×10 block form of V. Suppose $1 \le i \le 5 < j \le 10$ are such that $(i,j) \notin \{(6,2),(7,3),(10,1)\}$, and $V_{i,i}$ and $V_{j,j}$ have no common eigenvalue. Let $W = [W_{i,j}]_{i,j=1}^{10}$ be obtained from I (having the same size as V) by changing its (j,i) block $W_{j,i} = V_{j,i}(V_{j,j} - V_{i,i})^{-1}$. Here note that $V_{i,i}$ or $V_{j,j}$ is a scalar matrix. Then the (j,i) block of $W^{-1}VW$ is zero, and this transformation will not change other blocks in the matrix V. Hence, we can apply a number of such similarity transformations until we get all the desired zero blocks.

For part (b), let T be given by (6.2) with $\lambda \neq 0$. Then the Jordan form of T has Jordan blocks of size at most 2. This follows from a general fact that the Jordan form of a matrix

$$Z = \begin{bmatrix} \lambda I & 0 \\ Y & \lambda I \end{bmatrix}$$

consists of blocks of size at most 2, and the number of Jordan blocks of size 2 is equal to the rank of Y. (This fact is easily proven by using a rank decomposition $Y = W_1 \begin{bmatrix} I_{\operatorname{rank} Y} & 0 \\ 0 & 0 \end{bmatrix} W_2$, where W_1, W_2 are invertible.) Now, to prove (6.3), note that

$$r_1 + 2r_2 = \alpha + \gamma + x + y \qquad \text{and} \qquad s_1 + 2s_2 = \beta + \delta + x + y;$$

moreover, r_2 and s_2 are equal to the ranks of the matrices

$$\begin{array}{cc} \begin{array}{cc} \alpha & x \end{array} & \\ \begin{bmatrix} 0 & R_1 \\ R_2 & R_3 \end{bmatrix} & \begin{array}{c} \gamma \\ y \end{array} \end{array} \qquad \text{and} \qquad \begin{array}{cc} \begin{array}{cc} \beta & x \end{array} & \\ \begin{bmatrix} 0 & S_1 \\ S_2 & S_3 \end{bmatrix} & \begin{array}{c} \delta \\ y \end{array} \end{array},$$

respectively. Now,

$$\begin{aligned} r_2 &= \operatorname{rank}\begin{bmatrix} R_1 \\ R_3 \end{bmatrix} + \operatorname{rank}[R_2\ R_3] - \operatorname{rank} R_3 \\ &\le x + y - \operatorname{rank} R_3 \\ &\le x + y + \beta + \delta \\ &= s_1 + 2s_2. \end{aligned}$$

Similarly, one can prove that $s_2 \le r_1 + 2r_2$. □

We are now ready to describe the Jordan form of matrices $X \circ A$, where X is entrywise positive and A is an ICRN ray-pattern.

Theorem 6.5. *Let K be an $n \times n$ matrix in the Jordan form. Then K is similar to a matrix $X \circ A$, where X is entrywise positive and A is an ICRN ray-pattern, if and only if K has blocks of the following types only (perhaps after a permutation of the Jordan blocks of K):*

(1) $[\lambda]$, $\lambda \in \mathbb{R} \setminus \{0\}$;

(2)
$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \oplus [-\lambda], \qquad \lambda \in \mathbb{R} \setminus \{0\};$$

(3)
$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \oplus \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}, \qquad \lambda \in \mathbb{R} \setminus \{0\};$$

(4)
$$\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \oplus \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \oplus \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}, \qquad \lambda \in \mathbb{R} \setminus \{0\}.$$

(5)
$$\begin{bmatrix} \lambda & 0 \\ 0 & \overline{\lambda} \end{bmatrix} \oplus \begin{bmatrix} -\lambda & 0 \\ 0 & -\overline{\lambda} \end{bmatrix}, \qquad \lambda = \mu + i\nu, \quad 0 < \nu < \mu.$$

Proof. The "only if" part. By Theorem 4.2 and Lemma 6.3, we see that the Jordan form corresponding to nonreal eigenvalues has the form (5).

For a real eigenvalue λ, we can use Theorem 4.2 and Lemma 6.4 to conclude that the Jordan blocks corresponding to λ have the forms:

$$\text{(a)}\ \ [\lambda], \qquad \text{(b)}\ [-\lambda], \qquad \text{(c)}\ \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}, \qquad \text{(d)}\ \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix}$$

with multiplicities r_1, s_1, r_2, s_2, respectively, such that $r_1 + 2r_2 \ge s_2$ and $s_1 + 2s_2 \ge r_2$. Since $s_1 + 2s_2 \ge r_2$, we can use construct matrices of the form (4) and (2) until we use up all the Jordan blocks of the form (c). If we also used up the Jordan blocks of the form (d), then we are left with Jordan blocks of the forms (a) and (b), and we are done. So, suppose that we have not used up all the Jordan blocks of the form (d). Converting one matrix of the form in (4) to two matrices of the form (3), we can use up more Jordan blocks of the form (d). If we are not able to use up all the Jordan blocks of the form (d) after converting all type (4) matrices to type (3) matrices, then construct matrices of the form

$$(2') \quad \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} \oplus [\lambda],$$

which is a type (2) matrix with the roles of λ and $-\lambda$ interchanged. If this still do not exhaust all the Jordan blocks of the form (d), convert type (3) matrices to

$$(4') \quad \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} \oplus \begin{bmatrix} -\lambda & 1 \\ 0 & -\lambda \end{bmatrix} \oplus \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix},$$

which is a type (4) matrix with the roles of λ and $-\lambda$ interchanged. Since $r_1 + 2r_2 \geq s_2$, we will exhaust all the Jordan blocks of the form (d) through such conversions. Now, we see that all the Jordan blocks associated with λ and $-\lambda$ can be put in the forms (1), (2), (3), (4), (3′), (4′), and we are done.

The "if" part. For each of the matrices of the form (1) – (4), by Theorem 4.2 one can find a positive matrix X and an ICRN ray-pattern A such that $X \circ A$ have the following forms:

$$(1) \quad [\lambda], \qquad (2) \quad \lambda \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 2 & -1 & 1 \end{bmatrix},$$

$$(3) \quad \lambda \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 3 & 1 & 0 & 1 \\ -2 & -1 & 1 & 0 \end{bmatrix}, \qquad (4) \quad \lambda \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 2 & -3 & 1 & 0 & 0 \\ 4 & 5 & 6 & 0 & 0 & 1 \\ -1 & -1 & -1 & 0 & 1 & 0 \end{bmatrix},$$

and Lemma 6.3 shows that one can construct $X \circ A$ for case (5). A direct sum of the above matrices will give rise to a matrix with the desired Jordan form structure. □

As it follows from the proof of Theorem 6.5, conditions (1) - (4) of the theorem can be expressed also as follows: *The Jordan blocks corresponding to real eigenvalues of a matrix of the form $X \circ A$, where X is entrywise positive and A is an ICRN ray-pattern, are of size at most two, and if $r_1(\lambda)$ and $r_2(\lambda)$ are the numbers of the Jordan blocks of $X \circ A$ of sizes* 1 *and* 2*, respectively, corresponding to the real eigenvalue λ, then*

$$r_1(\lambda) + 2r_2(\lambda) \geq r_2(-\lambda). \tag{6.5}$$

Note that inequality (6.5) implies that if λ is an eigenvalue of $X \circ A$, but $-\lambda$ is not, then there are no Jordan blocks of size 2 corresponding to λ; indeed, apply (6.5) with λ replaced by $-\lambda$, and interpret $r_1(-\lambda)$ and $r_2(-\lambda)$ as zeros.

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