Norm Bounds on the Sum of Block Diagonal Matrices

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Abstract

We bound the norm of the sum of block diagonal matrices whose block structures may not be compatible, and use the result to bound the norm of banded positive semidefinite matrices. We consider the extension of the result to other norms.

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1 Introduction

Let $A, B \in M_n(\mathbf{F})$, where $\mathbf{F} = \mathbf{R}$ or $\mathbf{C}$. We have

$$\|A+B\| \le \|A\| + \|B\|, \tag{1.1}$$

where $\|\cdot\|$ denotes the (l_2) operator norm. If $A = A_1 \oplus A_2$ and $B = B_1 \oplus B_2$ with $A_1, B_1 \in M_p(\mathbf{F})$ for some $1 \le p < n$, then

$$\|A+B\| = \max\{\|A_1+B_1\|, \|A_2+B_2\|\} \le \max\{\|A_1\|+\|B_1\|, \|A_2\|+\|B_2\|\}. \tag{1.2}$$

However, if A and B have *incompatible* block structures, can one improve the estimate (1.1) by using the norms of the individual blocks of A and B? The purpose of this note is to answer this question and to discuss applications. In fact, inequality (1.2) also holds for other norms such as the numerical radius and Schur multiplier norms, and we discuss results on these norms.

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2 Main Results

Proposition 2.1 *Let $A, B \in M_n(\mathbf{F})$ be of the form $A = A_1 \oplus \cdots \oplus A_r$ and $B = B_1 \oplus \cdots \oplus B_s$, where $A_i \in M_{m_i}$, and $B_j \in M_{n_j}$. Set $D_A = \|A_1\| I_{m_1} \oplus \cdots \oplus \|A_r\| I_{m_r}$ and $D_B = \|B_1\| I_{n_1} \oplus \cdots \oplus \|B_s\| I_{n_s}$. Then*

$$\|A + B\| \leq \|D_A + D_B\| = \max\{(D_A + D_B)_{ii} : i = 1, \ldots, n\}.$$

Using the notation in Proposition 2.1, we say that A_i and B_j overlap if

$$1 + \sum_{k=1}^{j-1} n_k \leq 1 + \sum_{k=1}^{i-1} m_k \leq \sum_{k=1}^{j} n_k$$

or

$$1 + \sum_{k=1}^{i-1} m_k \leq 1 + \sum_{k=1}^{j-1} n_k \leq \sum_{k=1}^{i} m_k.$$

Imagining the blocks of A and B gives a better idea of what we mean by overlapping. Proposition 2.1 simply says:

$$\|A + B\| \leq \max\{\|A_i\| + \|B_j\| : A_i \text{ and } B_j \text{ overlap}\}. \tag{2.1}$$

The key to the proof is the observation [4, Lemma 3.5.12] that for any matrix X

$$\|X\| \leq t \quad \Leftrightarrow \quad \begin{pmatrix} tI & X \\ X^* & tI \end{pmatrix} \geq 0, \tag{2.2}$$

where the second inequality is in the positive semidefinite sense.

Proof. (of Proposition 2.1) Let A and B satisfy the hypotheses of the proposition. By (2.2), we have

$$0 \leq \begin{pmatrix} D_A & A \\ A^* & D_A \end{pmatrix}, \quad 0 \leq \begin{pmatrix} D_B & B \\ B^* & D_B \end{pmatrix},$$

and hence

$$0 \leq \begin{pmatrix} D_A + D_B & (A + B) \\ (A + B)^* & D_A + D_B \end{pmatrix} \leq \begin{pmatrix} \|D_A + D_B\| I & (A + B) \\ (A + B)^* & \|D_A + D_B\| I \end{pmatrix}.$$

By (2.2) again, we have

$$\|D_A + D_B\| \geq \|A + B\|,$$

as desired. □

This result can be extended to more that two matrices using exactly the same proof. We have not presented the general result nor the general proof as the necessary notation would obscure the simplicity of the proof. Despite its simplicity Proposition 2.1 yields eigenvalue bounds that are not immediately obvious. The following corollary solves a problem posed by Parlett in the American Mathematical Monthly [8].

Corollary 2.2 *Suppose A is a positive semidefinite matrix with diagonal entries $d_1, \ldots, d_n$, such that the (i,j) entries are zero for all $|i-j| > r$ for some positive integer r. Then the eigenvalues lie in the interval $[0, d]$ where*

$$d = \max\{d_i + \cdots + d_{i+r} : i = 1, \ldots, n-r\}. \tag{2.3}$$

Proof. Let $A = LL^*$ be a Cholesky factorization of A. Since A is positive semidefinite it is sufficient to show that $\|A\| \le d$. Then

$$\|A\| = \|LL^*\| = \|L^*L\|.$$

Let L^* have columns $v_1, \ldots, v_n$. Then

$$L^*L = \sum_{i=1}^{n} v_i v_i^*.$$

By the banded structure of A the vector v_i has zero entries except possibly in the positions $j \equiv \max\{1, i-r\}, j+1, \ldots, i$. Thus, $v_i v_i^*$ is a block diagonal matrix, and its norm is

$$\|v_i\|^2 = v_i^* v_i = d_i.$$

For each matrix $v_i v_i^*$, define $D_{v_i v_i^*}$ as in Proposition 2.1. The jth diagonal entry of $\sum_{i=1}^n D_{v_i v_i^*}$ is $d_i + \cdots + d_{\min\{i+r,n\}}$. The extended version of Proposition 2.1 now yields $\|A\| \le d$. □

A correlation matrix A necessarily has entries of modulus at most 1. By Gershgorin's Theorem (see [4, Theorem 6.1.1]), if S is a banded correlation matrix, with (i,j) entry equal to zero for $|i-j| > r$, then all the eigenvalues of A lie in $[0, 2r-1]$, and hence $\|A\| \le 2r-1$. The specialization of Corollary 2.2 says that in fact, $\|A\| \le r+1$.

Corollary 2.3 *Suppose A is a correlation matrix such that the (i,j) entries are zero for all $|i-j| > r$ for some positive integer r, then the eigenvalues lie in the interval $[0, r+1]$.*

The bound in Corollary 2.2 is tight, and hence so is that in Corollary 2.3: Given any pair r, n with $r < n$, and any nonnegative numbers $d_1, \ldots, d_n$ we can construct a positive semidefinite matrix A with $a_{ij} = 0$ for $|i-j| > r$, diagonal entries $d_1, \ldots, d_n$, and largest eigenvalue equal to the quantity d in (2.3). Let $i \le n-r$ be such that $d_i + \cdots + d_{i+r} = d$, and let $D = \mathrm{diag}(d_1^{1/2}, \ldots, d_n^{1/2})$. The bound in Corollary 2.2 is tight for the matrix

$$A = D(I_{i-1} \oplus J_{r+1} \oplus I_{n-(r+i)})D,$$

where J_k is the $k \times k$ matrix of ones. Note that the principal submatrix of A in rows and columns $i, i+1, \ldots, i+r$ has rank 1, and hence its trace, which is $d_i + \cdots + d_{i+r}$, is an eigenvalue.

For a given banded matrix $A \in M_n(\mathbf{F})$, one may consider its LU factorization, and use an argument like that in Corollary 2.2 to estimate the norm of A. However, unless the matrix A admits an easy LU factorization, the procedure and the statement of the result will not be as clean as that for positive semidefinite matrices.

3 Block Matrix Norms

In this section, we consider the extension of Proposition 2.1 to other types of norms on $M_n(\mathbf{F})$. Note that the key to the proof of Proposition 2.1 was the characterization (2.2) of the operator norm in terms of the positive semidefiniteness of a block matrix. There are other norms that can be characterized in this way and so one can prove analogs of Proposition 2.1 for these norms. To simplify the notation we shall restrict our attention to square matrices.

The *numerical radius* of $A \in M_n$ is defined by

$$\omega(A) = \max\{|x^*Ax| : x \in \mathbf{F}^n,\ x^*x = 1\}.$$

Ando [1] has shown that $\omega(A) \leq t$ if and only if there is a Hermitian matrix Z such that

$$\begin{pmatrix} tI + Z & A \\ A^* & tI - Z \end{pmatrix} \geq 0$$

in the positive semidefinite sense. The *Hadamard operator norm* of a matrix A is defined by

$$\|A\|_H \equiv \max\{\|A \circ B\| : \|B\| \leq 1\}$$

where $\circ$ denotes Hadamard or entrywise multiplication. Haagerup (unpublished) showed[1] that $\|A\|_H \leq t$ if and only if there are matrices Hermitian P and Q with $p_{ii} = q_{ii} = 0$, $i = 1, \ldots, n$, such that

$$\begin{pmatrix} tI + P & A \\ A^* & tI + Q \end{pmatrix} \geq 0.$$

One may also consider the norm of Hadamard multiplication with respect to the numerical radius:

$$\|A\|_{H,\omega} \equiv \max\{w(A \circ B) : w(B) \leq 1\}.$$

Ando and Okubo [2] have shown that $\|A\|_{H,\omega} \leq t$ if and only if there is a Hermitian matrix P with $p_{ii} = 0$, $i = 1, \ldots, n$, such that

$$\begin{pmatrix} tI + P & A \\ A^* & tI + P \end{pmatrix} \geq 0.$$

See [6] for a unified approach to the proofs of both these representations and for references to other proofs.

Let $N(\cdot)$ be a norm on matrices of any size. We say that $N(\cdot)$ is an $\mathcal{S}$-norm if for each positive integer n, there exists a real subspace $\mathcal{S}_n \subset M_n(\mathbf{F}) \times M_n(\mathbf{F})$ of complex Hermitian or real symmetric matrix pairs (depending on $\mathbf{F} = \mathbf{C}$ or $\mathbf{R}$) so that $A \in M_n$ satisfies $N(A) \leq t$ if and only if there exists $(P, Q) \in \mathcal{S}_n$ satisfying

$$\begin{pmatrix} tI + P & A \\ A^* & tI + Q \end{pmatrix} \geq 0.$$

[1] Actually, Haagerup, Ando and Okubo [2], and Mathias [6] stated their results in a slightly different form, but it is easy to show that their results are indeed equivalent to the results that we assert in this paragraph.

The norms we have mentioned are all $\mathcal{S}$-norms. For the operator norm take

$$\mathcal{S}_n = \{(0_n, 0_n)\},$$

for the numerical radius take

$$\mathcal{S}_n = \{(Z, -Z) : Z \in H_n\},$$

for $\|\cdot\|_H$ take

$$\mathcal{S}_n = \{(P, Q) : p_{ii} = q_{ii} = 0 \ \ i = 1, \ldots, n\}$$

and for $\|\cdot\|_{H,\omega}$ take

$$\mathcal{S}_n = \{(P, P) : p_{ii} = 0 \ \ i = 1, \ldots, n\}.$$

In each case the family $\{\mathcal{S}_n\}_{n=1}^{\infty}$ has the following *direct sum property*:

$$(P_1, Q_1) \in \mathcal{S}_{n_1}, \ \ (P_2, Q_2) \in \mathcal{S}_{n_2} \quad \Longrightarrow \quad (P_1 \oplus P_2, Q_1 \oplus Q_2) \in \mathcal{S}_{n_1+n_2}.$$

For simplicity, we say that the $\mathcal{S}$-norm has the direct sum property if the above property is satisfied. We have the following generalization of Proposition 2.1.

Proposition 3.1 *Let $N(\cdot)$ be an $\mathcal{S}$-norm with the direct sum property. Suppose $A, B \in M_n(\mathbf{F})$ are of the form $A = A_1 \oplus \cdots \oplus A_r$ and $B = B_1 \oplus \cdots \oplus B_s$, where $A_i \in M_{m_i}$, and $B_j \in M_{n_j}$. Let $a_i = N(A_i)$ and $b_j = N(B_j)$. Set $D_A = a_1 I_{m_1} \oplus \cdots \oplus a_r I_{m_r}$ and $D_B = b_1 I_{n_1} \oplus \cdots \oplus b_s I_{n_s}$. Then*

$$N(A+B) \le N(D_A + D_B) = \max\{N(A_i) + N(B_j) : A_i \text{ and } B_j \text{ overlap}\}. \tag{3.1}$$

Proof. Since $N(\cdot)$ is an $\mathcal{S}$-norm, there are pairs $(P_i, Q_i) \in \mathcal{S}_{m_i}$ and $(R_j, S_j) \in \mathcal{S}_{n_j}$ such that

$$\begin{pmatrix} a_i I + P_i & A_i \\ A_i^* & a_i I + Q_i \end{pmatrix} \ge 0, \ \text{ and } \ \begin{pmatrix} b_j I + R_j & B_j \\ B_j^* & b_j I + S_j \end{pmatrix} \ge 0.$$

Taking a direct sum and performing a block permutation we have

$$\begin{pmatrix} D_A + P & A \\ A^* & D_A + Q \end{pmatrix} \ge 0, \ \text{ and } \ \begin{pmatrix} D_B + R & B \\ B^* & D_B + S \end{pmatrix} \ge 0, \tag{3.2}$$

where

$$P = P_1 \oplus \cdots \oplus P_r, \ Q = Q_1 \oplus \cdots \oplus Q_r, \ R = R_1 \oplus \cdots \oplus R_s, \ S = S_1 \oplus \cdots \oplus S_s.$$

By the direct sum property (P, Q) and (R, S) are both in $\mathcal{S}_n$, and since $\mathcal{S}_n$ is a subspace, $(P + R, Q + S)$ is also in $\mathcal{S}_n$. Adding (3.2) we have

$$\begin{pmatrix} (D_A + D_B) + (P + R) & (A + B) \\ (A + B)^* & (D_A + D_B) + (Q + S) \end{pmatrix} \ge 0.$$

Let

$$d \equiv N(D_A + D_B) = \max\{N(A_i) + N(B_j) : A_i \text{ and } B_j \text{ overlap}\}.$$

Then $D_A + D_B \le dI$ so

$$\begin{pmatrix} dI + (P+R) & (A+B) \\ (A+B)^* & dI + (Q+S) \end{pmatrix} \ge 0.$$

We have seen that $(P+R, Q+S) \in \mathcal{S}_n$, so, since $N(\cdot)$ is an $\mathcal{S}$-norm, we have the desired bound $N(A+B) \le d$. □

4 Further Results and Questions

A class of norms on $M_n(\mathbf{F})$, which is used frequently, is the class of *unitarily invariant norms*, i.e., those norms $N(\cdot)$ satisfying $N(UAV) = N(A)$ for all $A, B, U, V \in M_n(\mathbf{F})$ such that $U^*U = V^*V = I$. This class of norms includes the Ky Fan k-norms and the Schatten p-norms. The operator norm $\|\cdot\|$ is in the intersection of these two classes of norms. See [3, Example 7.4.54] for further discussion and a simple characterization of these norms. We shall, as far as possible, generalize Proposition 2.1 to unitarily invariant norms. To do this, we need the following lemma which is [4, Theorem 3.5.15] (with $p = 1$).

Lemma 4.1 *Let $N(\cdot)$ be a unitarily invariant norm. If $C, D_1, D_2 \in M_n(\mathbf{F})$ are such that*

$$\begin{pmatrix} D_1 & C \\ C^* & D_2 \end{pmatrix} \ge 0 \tag{4.1}$$

then

$$N^2(C) \le N(D_1)N(D_2).$$

Proposition 4.2 *Let $N(\cdot)$ be a unitarily invariant norm on $M_n(\mathbf{F})$. Suppose $A, B \in M_n(\mathbf{F})$ are of the form $A = A_1 \oplus \cdots \oplus A_r$ and $B = B_1 \oplus \cdots \oplus B_s$, where $A_i \in M_{m_i}$, and $B_j \in M_{n_j}$. Set $D_A = \|A_1\| I_{m_1} \oplus \cdots \oplus \|A_r\| I_{m_r}$ and $D_B = \|B_1\| I_{n_1} \oplus \cdots \oplus \|B_s\| I_{n_s}$. Then*

$$N(A+B) \le N(D_A + D_B). \tag{4.2}$$

Proof. Let $C = A + B$ and $D_1 = D_2 = D_A + D_B$. The result follows from Lemma 4.1. □

Note that if $N(\cdot)$ is not the operator norm then $N(D_A + D_B)$ is not necessarily equal to $\max\{|(D_A + D_B)_{ii}| : i = 1, \ldots, n\}$ as in Proposition 2.1. In fact, it follows from [4, Theorem 3.5.18] that a unitarily invariant norm $N(\cdot)$ on $M_n(\mathbf{F})$ satisfies $N(D) = \max\{|D_{ii}| : i = 1, \ldots, n\}$ for any diagonal matrix D if and only if $N(\cdot)$ is the operator norm.

We presented Proposition 4.2 as a generalization of Proposition 2.1. If one wants a stronger bound on $N(A+B)$ one can use the fact that the matrix in (4.1) is positive semidefinite if one takes $D_1 = (CC^*)^{1/2}$ and $D_2 = (C^*C)^{1/2}$ to show that (4.2) is still true with $D_A = (A_1A_1^*)^{1/2} \oplus \cdots \oplus (A_rA_r^*)^{1/2}$ and $D_B = (B_1B_1^*)^{1/2} \oplus \cdots \oplus (B_sB_s^*)^{1/2}$.

A norm $N(\cdot)$ is a *unitary similarity invariant* norm (or a *weakly unitarily invariant* norm) if $N(U^*AU) = N(A)$ for any $A, U \in M_n(\mathbf{F})$ with $U^*U = I$. Evidently, every unitarily invariant norm is a unitary similarity invariant norm. The numerical radius is an example of unitary similarity invariant norm, which is not unitarily invariant. One may wonder whether (4.2) is valid for all unitary similarity invariant norms. It is not:

Example 4.3 Define $N(\cdot)$ on $M_2(\mathbf{F})$ by

$$N(A) = \max\{|3x^*Ax - \operatorname{tr} A| : x \in \mathbf{F}^2,\ x^*x = 1\}.$$

Then (4.2) fails for $A = \operatorname{diag}(1,0)$ and $B = \operatorname{diag}(0,-1)$ as $N(A+B) = 3 > 1 = N(D_A+D_B)$. It would be interesting to characterize the unitary similarity invariant norms that satisfy (4.2).

Next, we consider norms $N(\cdot)$ on $M_n(\mathbf{F})$ induced by vector norms ν on $\mathbf{F}^n$, i.e.,

$$N(A) = \max\{\nu(Ax) : x \in \mathbf{F}^n,\ \nu(x) \le 1\}.$$

The usual operator norm $\|\cdot\|$ is the norm induced by the l_2 norm on $\mathbf{F}^n$. The column sum norm on $M_n(\mathbf{F})$ defined by

$$N_c(A) = \max_{1\le j\le n} \sum_{i=1}^{n} |A_{ij}|$$

is induced by the l_1 norm on $\mathbf{F}^n$, and the row sum norm defined by

$$N_r(A) = \max_{1\le i\le n} \sum_{j=1}^{n} |A_{ij}|$$

is induced by the l_∞ norm on $\mathbf{F}^n$. One readily checks that the conclusion of Proposition 2.1 also holds for these two norms. It would be interesting to extend the result to norms induced by the l_p norms for $p \ne 1, 2, \infty$.

More generally, one may ask whether Proposition 2.1 can be extended to norms $N(\cdot)$ on $M_n(\mathbf{F})$ induced by other norms ν on $\mathbf{F}^n$. It is known (e.g., see [7, Theorem 1]) that if $N(\cdot)$ is induced by an absolute norm ν, i.e., $\nu(x) = \nu(Px)$ for all diagonal matrix P satisfying $P^*P = I$, then

$$N(D) = \max\{|D_{ii}| : 1 \le i \le n\},$$

for any diagonal matrix D. It is also known (e.g., see [7, Theorems 5 and 6]) that if $n > 2$ and $N(\cdot)$ is induced by a norm ν on $\mathbf{F}^n$ that is not a multiple of an l_p norm, then one can find matrices $A_1 \in M_m$ and $B_2 \in M_{n-m}$ for which

$$N(A_1 \oplus B_2) > \max\{N(A_1), N(B_2)\}.$$

Taking $A = A_1 \oplus 0_{n-m}$ and $B = 0_m \oplus B_2$ we have

$$N(A+B) = N(A \oplus B) > \max\{N(A_1), N(B_2)\} = N(D_A + D_B);$$

so an extension of Proposition 2.1 is impossible.

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References

[1] T. Ando. On the structure of operators with numerical radius one, *Acta Sci. Math.*, 34:11–15, 1973.

[2] T. Ando and K. Okubo. Induced norms of the Schur multiplier operator, *Linear Algebra Appl.*, 147:181–199, 1991.

[3] R. A. Horn and C. R. Johnson. *Matrix Analysis*, Cambridge University Press, New York, 1985.

[4] R. A. Horn and C. R. Johnson. *Topics in Matrix Analysis*, Cambridge University Press, New York, 1991.

[5] A.W. Marshall and I. Olkin. *Inequalities: The Theory of Majorization and its Applications,* Academic Press, New York, 1979.

[6] R. Mathias. Matrix completions, Hadamard products and norms. *Proc. Amer. Math. Soc.*, 117(4):905–918, 1993.

[7] J.K. Merikoski. Operator norms of submatrices, *Linear Algebra Appl.* 36: 173-188, 1981.

[8] B. N. Parlett. Problem 10839. *Amer. Math. Monthly*, 107(10):950, Dec. 2000.